Binomial Expansions as Infinite Sums

Unfortunately, you only get a nice, neat, finite expansion when you've got a positive integer n.
But that pesky n sometimes likes to be a negative number or a fraction. n for nuisance. n for naughty.

If ***n is Negative*** *or a* ***Fraction*** *the expansion is an* ***Infinite Sum***

You'll be happy to know (possibly?) that you can expand $(1 + x)^n$ even when n **isn't** a positive integer.
The formula is very similar to the one on p.50:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times\ldots\times r}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

This bit is just like the expansion in the example on p.50.

This means that n is a 'real number', which basically means any number — negative, fraction, irrational etc.

If n isn't a positive integer, this expansion is an **infinite sum** (if n is a positive integer, all the terms with $r > n$ are zero, and you get the formula from p.50). It's only valid for $|x| < 1$ **—** otherwise you get a **divergent series** (see p.48).

Because the expansion is infinite, a question will usually tell you **how many terms** to work out.

Example: Find the binomial expansion of $\frac{1}{(1+x)^2}$ up to and including the term in x^3.

First, **rewrite the expression:** $\frac{1}{(1+x)^2} = (1 + x)^{-2}$.

Now use the **general formula**. Here, $n = -2$:

$$(1+x)^{-2} = 1 + (-2)x + \frac{(-2)\times(-2-1)}{1\times 2}x^2 + \frac{(-2)\times(-2-1)\times(-2-2)}{1\times 2\times 3}x^3 + \ldots$$

$$= 1 + (-2)x + \frac{(-2)\times(-3)}{1\times 2}x^2 + \frac{(-2)\times(-3)\times(-4)}{1\times 2\times 3}x^3 + \ldots$$

$$= 1 + (-2)x + \frac{3}{1}x^2 + \frac{-4}{1}x^3 + \ldots$$

$$= \mathbf{1 - 2x + 3x^2 - 4x^3 + \ldots}$$

(n = −2; n(n − 1))

You can cancel down before you multiply — but be careful with those minus signs.

With a negative n, you'll never get zero as a coefficient. If the question hadn't told you to stop, the expansion could go on forever.

We've left out all the terms after $-4x^3$, so the cubic equation you've ended up with is an **approximation** to the original expression. You could also write the answer like this: $\frac{1}{(1+x)^2} \approx 1 - 2x + 3x^2 - 4x^3$.

Because the formula is in the form $(1 + x)^n$, you might need to do a bit of factorising first...

Example: Find the binomial expansion of $\sqrt{4+3x}$ up to and including the term in x^3.

This time we've got a **fractional power:** $\sqrt{4+3x} = (4+3x)^{\frac{1}{2}}$

You need the first term to be 1, so take out a factor of 2:

$$(4+3x)^{\frac{1}{2}} = \left(4\left(1+\frac{3}{4}x\right)\right)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{3}{4}x\right)^{\frac{1}{2}} = 2\left(1+\frac{3}{4}x\right)^{\frac{1}{2}}$$

So this time $n = \frac{1}{2}$, and you also need to replace x with $\frac{3}{4}x$:

$$\left(1+\frac{3}{4}x\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{3}{4}x\right) + \frac{\frac{1}{2}\times\left(\frac{1}{2}-1\right)}{1\times 2}\left(\frac{3}{4}x\right)^2 + \frac{\frac{1}{2}\times\left(\frac{1}{2}-1\right)\times\left(\frac{1}{2}-2\right)}{1\times 2\times 3}\left(\frac{3}{4}x\right)^3 + \ldots$$

$$= 1 + \frac{3}{8}x + \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)}{2}\,\frac{9}{16}x^2 + \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)}{6}\,\frac{27}{64}x^3 + \ldots$$

$$= 1 + \frac{3}{8}x + \left(-\frac{1}{8}\times\frac{9}{16}\right)x^2 + \left(\frac{3}{48}\times\frac{27}{64}\right)x^3 + \ldots$$

$$= 1 + \frac{3}{8}x - \frac{9}{128}x^2 + \frac{27}{1024}x^3 + \ldots$$

($n = \frac{1}{2}$; n(n − 1))

So $(4+3x)^{\frac{1}{2}} = 2\left(1+\frac{3}{4}x\right)^{\frac{1}{2}}$

$$= 2\left(1 + \frac{3}{8}x - \frac{9}{128}x^2 + \frac{27}{1024}x^3 + \ldots\right) = \mathbf{2 + \frac{3}{4}x - \frac{9}{64}x^2 + \frac{27}{512}x^3 + \ldots}$$

Gus didn't have any friends, so he didn't even qualify for the standard 'bi-gnome-ial' joke.

Cancelling down is much trickier with this type of expansion — it's usually safer to multiply everything out fully.

Binomial Expansions as Infinite Sums

Some **Binomial Expansions** are only **Valid** for **Certain Values** of x

When you find a binomial expansion, you usually have to state which values of x the expansion is valid for.

If n is a **positive integer**, the binomial expansion of $(p + qx)^n$ is valid for **all values of x**.
If n is a **negative integer** or a **fraction**, the binomial expansion of $(p + qx)^n$ is valid when $\left|\frac{qx}{p}\right| < 1$ (or $|x| < \left|\frac{p}{q}\right|$).

So, in the examples on the previous page:

$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
This expansion is valid for $|x| < 1$.

$(1 + 2x)^{\frac{1}{3}} = 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 + \ldots$
This expansion is valid if $|2x| < 1 \Rightarrow 2|x| < 1 \Rightarrow |x| < \frac{1}{2}$.

You can use the formula on a **Combination** of expansions

Some nasty expressions can be expanded by doing two or more simple expansions and multiplying them together.

Example: Find the first three terms in the expansion of $\frac{(1+2x)^3}{(1-x)^2}$, stating the range of x for which it is valid.

First re-write the expression as a **product** of two expansions: $\frac{(1+2x)^3}{(1-x)^2} = (1 + 2x)^3(1 - x)^{-2}$

Expand each of these separately using the **formula**:

$$(1 + 2x)^3 = 1 + 3(2x) + \frac{3\times 2}{1\times 2}(2x)^2 + \frac{3\times 2\times 1}{1\times 2\times 3}(2x)^3 = 1 + 6x + 3(4x^2) + 8x^3 = \mathbf{1 + 6x + 12x^2 + 8x^3}$$

$$(1 - x)^{-2} = 1 + (-2)(-x) + \frac{(-2)\times(-3)}{1\times 2}(-x)^2 + \frac{(-2)\times(-3)\times(-4)}{1\times 2\times 3}(-x)^3 + \ldots = \mathbf{1 + 2x + 3x^2 + 4x^3 + \ldots}$$

Multiply the two expansions together. Since you're only asked for the **first three terms**, ignore any terms with **higher powers** of x than x^2.

$$(1 + 2x)^3(1 - x)^{-2} = (1 + 6x + 12x^2 + 8x^3)(1 + 2x + 3x^2 + 4x^3 + \ldots)$$
$$= 1(1 + 2x + 3x^2) + 6x(1 + 2x) + 12x^2(1) + \ldots$$
$$= 1 + 2x + 3x^2 + 6x + 12x^2 + 12x^2 + \ldots = \mathbf{1 + 8x + 27x^2 + \ldots}$$

Now find the **validity** of each expansion: $(1 + 2x)^3$ is valid for **all values** of x, since n is a **positive integer**.
$(1 - x)^{-2}$ is valid if $|-x| < 1 \Rightarrow |x| < 1$.

So the expansion of $\frac{(1+2x)^3}{(1-x)^2}$ is only valid if $|x| < 1$.

*For the **combined** expansion to be **valid**, x must be in the valid range for **both** expansions, i.e. the **narrowest** of the valid ranges.*

Practice Questions

Q1 Find the binomial expansion of each of the following, up to and including the term in x^3:

a) $\frac{1}{(1+x)^4}$ b) $\frac{1}{(1-3x)^3}$ c) $\sqrt{1-5x}$

Q2 If the binomial expansion of $(4 - 2x)^n$ is an infinite series, what values of x is the expansion valid for?

Q3 Give the binomial expansions of the following, up to and including the term in x^2.
State which values of x each expansion is valid for. a) $\frac{1}{(3+2x)^2}$ b) $\sqrt[3]{8-x}$

Exam Question

Q1 a) Find the binomial expansion of $\frac{1}{\sqrt{9-4x}}$, up to and including the term in x^3. [5 marks]

b) Hence find the first three terms in the expansion of $\frac{2-x}{\sqrt{9-4x}}$. [4 marks]

This is by-no-means the last page on binomials...

I'm afraid that the formula from p.51 won't help with these expansions — if it's not a 1 in the brackets, you'll have to factorise before using the formula (and make sure to raise the factor to the negative or fractional power as well).

Further Binomial Expansions

Binomial expansions on their own are pretty nifty, but when you combine them with partial fractions (see p.14-15) they become all-powerful. I'm sure there's some sort of message about friendship or something in there...

Split *functions into* ***Partial Fractions****, then add the* ***Expansions***

You can find the binomial expansion of even more complicated functions by splitting them into partial fractions first.

Example: $f(x) = \dfrac{x-1}{(3+x)(1-5x)}$

a) f(x) can be expressed in the form $\dfrac{A}{(3+x)} + \dfrac{B}{(1-5x)}$. Find the values of A and B.

b) Use your answer to part a) to find the binomial expansion of f(x) up to and including the term in x^2.

c) Find the range of values of x for which your answer to part b) is valid.

a) Convert f(x) into **partial fractions**:

$$\frac{x-1}{(3+x)(1-5x)} \equiv \frac{A}{(3+x)} + \frac{B}{(1-5x)} \Rightarrow x - 1 \equiv A(1-5x) + B(3+x)$$

Let $x = -3$, then $-3 - 1 = A(1 - (-15)) \Rightarrow -4 = 16A \Rightarrow A = -\frac{1}{4}$

Let $x = \frac{1}{5}$, then $\frac{1}{5} - 1 = B\left(3 + \frac{1}{5}\right) \Rightarrow -\frac{4}{5} = \frac{16}{5}B \Rightarrow B = -\frac{1}{4}$

See p.14-15 if you need a reminder about how to do partial fractions.

b) Start by **rewriting** the partial fractions in $(a + bx)^n$ form:

$$f(x) = -\frac{1}{4}(3+x)^{-1} - \frac{1}{4}(1-5x)^{-1}$$

Now do the two **binomial expansions**:

$$(3+x)^{-1} = \left(3\left(1 + \frac{1}{3}x\right)\right)^{-1}$$
$$= \frac{1}{3}\left(1 + \frac{1}{3}x\right)^{-1}$$
$$= \frac{1}{3}\left(1 + (-1)\left(\frac{1}{3}x\right) + \frac{(-1)(-2)}{2}\left(\frac{1}{3}x\right)^2 + \dots\right)$$
$$= \frac{1}{3}\left(1 - \frac{1}{3}x + \frac{1}{9}x^2 + \dots\right)$$
$$= \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 + \dots$$

$$(1-5x)^{-1} = 1 + (-1)(-5x) + \frac{(-1)(-2)}{2}(-5x)^2 + \dots$$
$$= 1 + 5x + 25x^2 + \dots$$

And put **everything together**: $f(x) = -\frac{1}{4}(3+x)^{-1} - \frac{1}{4}(1-5x)^{-1}$

$$\approx -\frac{1}{4}\left(\frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2\right) - \frac{1}{4}(1 + 5x + 25x^2)$$
$$= -\frac{1}{12} + \frac{1}{36}x - \frac{1}{108}x^2 - \frac{1}{4} - \frac{5}{4}x - \frac{25}{4}x^2$$
$$= -\frac{1}{3} - \frac{11}{9}x - \frac{169}{27}x^2$$

c) Each of the two expansions from part b) is valid for different values of x.
The combined expansion of f(x) is valid where these two ranges **overlap**, i.e. over the **narrower of the two ranges**.

The expansion of $(3+x)^{-1}$ is valid when $\left|\frac{x}{3}\right| < 1 \Rightarrow \frac{|x|}{|3|} < 1 \Rightarrow |x| < 3$.

The expansion of $(1-5x)^{-1}$ is valid when $|-5x| < 1 \Rightarrow |-5||x| < 1 \Rightarrow |x| < \frac{1}{5}$.

The expansion of f(x) is valid for values of x in both ranges,
so the expansion of f(x) is valid for $|x| < \frac{1}{5}$.

Remember — the expansion of $(p + qx)^n$ is valid when $\left|\frac{qx}{p}\right| < 1$.

You might already know the rules $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.
If you don't, then get to know them — they're handy for rearranging these limits.

Further Binomial Expansions

*To find **Approximations**, substitute the right value of **x***

When you've done an **expansion**, you can use it to **estimate** the value of the original expression for given values of x.

Example: The binomial expansion of $(1+3x)^{\frac{1}{3}}$ up to the term in x^3 is $(1+3x)^{\frac{1}{3}} \approx 1 + x - x^2 + \frac{5}{3}x^3$.
The expansion is valid for $|x| < \frac{1}{3}$.
Use this expansion to approximate $\sqrt[3]{1.3}$. Give your answer to 4 d.p.

For this type of question, you need to find **the right value of x** to make the expression you're expanding equal to the thing you're looking for.

In this case it's pretty straightforward: $\sqrt[3]{1.3} = (1+3x)^{\frac{1}{3}}$ when $x = 0.1$

$$\sqrt[3]{1.3} = (1+3(0.1))^{\frac{1}{3}}$$
$$\approx 1 + 0.1 - (0.1)^2 + \frac{5}{3}(0.1)^3$$
$$= 1 + 0.1 - 0.01 + \frac{0.005}{3}$$
$$= \mathbf{1.0917} \text{ (4 d.p.)}$$

$0.1 < \frac{1}{3}$, so the expansion is valid for this approximation.

Don't forget to use a "≈" here — the answer's an approximation because you're only using the expansion up to the x^3 term.

This is the expansion given in the question, with x = 0.1.

In **trickier cases** you have to do a spot of **rearranging** to get to the answer.

Example: The binomial expansion of $(1-5x)^{\frac{1}{2}}$ up to the term in x^2 is $(1-5x)^{\frac{1}{2}} \approx 1 - \frac{5x}{2} - \frac{25}{8}x^2$ for $|x| < \frac{1}{5}$.
Use $x = \frac{1}{50}$ in this expansion to find an approximate value for $\sqrt{10}$, and find the percentage error in your approximation, giving your answer to 2 significant figures.

First, sub $x = \frac{1}{50}$ into **both sides** of the expansion:

$$\sqrt{\left(1 - 5\left(\frac{1}{50}\right)\right)} \approx 1 - \frac{5}{2}\left(\frac{1}{50}\right) - \frac{25}{8}\left(\frac{1}{50}\right)^2$$
$$\sqrt{\left(1 - \frac{1}{10}\right)} \approx 1 - \frac{1}{20} - \frac{1}{800}$$
$$\sqrt{\frac{9}{10}} \approx \frac{759}{800}$$

Now **simplify** the square root and **rearrange** to find an estimate for $\sqrt{10}$:

$$\sqrt{\frac{9}{10}} = \frac{\sqrt{9}}{\sqrt{10}} = \frac{3}{\sqrt{10}} \approx \frac{759}{800} \Rightarrow \sqrt{10} \approx 3 \div \frac{759}{800} = \mathbf{\frac{800}{253}}$$

The **percentage error** is $\left|\frac{\text{real value} - \text{estimate}}{\text{real value}}\right| \times 100 = \left|\frac{\sqrt{10} - \frac{800}{253}}{\sqrt{10}}\right| \times 100 = \mathbf{0.0070\%}$ (2 s.f.)

So this is a pretty darn good estimate.

Practice Questions

Q1 Given that $f(x) = \frac{2x-7}{(x+1)(x-2)} \equiv \frac{3}{x+1} - \frac{1}{x-2}$, find the binomial expansion of $f(x)$ up to and including the x^2 term.

Q2 Use the approximation $\sqrt{\frac{1+2x}{1-3x}} \approx 1 + \frac{5}{2}x + \frac{35}{8}x^2$ with $x = \frac{2}{15}$ to show that $\sqrt{19} \approx \frac{127}{30}$.

Exam Questions

Q1 a) Find the binomial expansion of $(16+3x)^{\frac{1}{4}}$, for $|x| < \frac{16}{3}$, up to and including the term in x^2. [5 marks]

b) (i) Estimate $\sqrt[4]{12.4}$ by substituting a suitable value of x into your expansion from part (a). Give your answer to 6 decimal places. [2 marks]

(ii) What is the percentage error in this estimate? Give your answer to 3 significant figures. [2 marks]

Q2 Find the binomial expansion, up to the term in x^2, of $f(x) = \frac{13x-17}{(5-3x)(2x-1)}$ for $|x| < \frac{5}{3}$. [9 marks]

Don't mess with me — I'm a partial arts expert...

You can also use the binomial expansion to estimate the values of fractions — for example, the binomial expansion of $(1+3x)^{-1}$ can be used to estimate $\frac{100}{103}$. Just write $\frac{100}{103}$ as $\frac{1}{1+0.03}$, then substitute $x = 0.01$ into the expansion.