Sequences

A sequence is a list of numbers that follow a certain pattern. Sequences can be finite or infinite (infinity — oooh), and there are a couple of types you need to know about. And guess what? You have to know everything about them.

A **Sequence** can be defined by its **n^{th} Term**

You almost definitely covered this stuff at GCSE, so **no excuses** for mucking it up.
The point of all this is to show how you can work out any **value** (**the n^{th} term**) from its **position** in the sequence (n).

Example: Find the n^{th} term of the sequence 5, 8, 11, 14, 17, ...

1st	2nd	3rd	4th	5th
5	8	11	14	17

+3 +3 +3 +3

Each term is **3 more** than the one before it. That means that you need to start by **multiplying n by 3**.

Take the first term (where $n = 1$). If you multiply n by 3, you still have to **add 2** to get 5.

The same goes for $n = 2$. To get 8 you need to multiply n by 3, then add 2.
Every term in the sequence is worked out exactly the same way. So the n^{th} term is $3n + 2$.

You can define a sequence by a ***Recurrence Relation*** *too*

Don't be put off by the fancy name — recurrence relations are pretty **easy** really.

> The main thing to remember is:
> **u_n just means the n^{th} term of the sequence**

The **next term** in the sequence is u_{n+1}. You need to describe how to **work out** u_{n+1} if you're given u_n.

Example: Find the recurrence relation of the sequence 5, 8, 11, 14, 17, ...

From the example above, you know that each term equals the one before it, plus 3.

This is written like this: $u_{n+1} = u_n + 3$

So, if $n = 5$, $u_n = u_5$ which stands for the 5th term, and $u_{n+1} = u_6$ which stands for the 6th term.

In everyday language, $u_{n+1} = u_n + 3$ means that the sixth term equals the fifth term plus 3.

BUT $u_{n+1} = u_n + 3$ on its own **isn't enough** to describe 5, 8, 11, 14, 17, ...

For example, the sequence 87, 90, 93, 96, 99, ... **also** has each term being 3 more than the one before.

The description needs to be more **specific**, so you've got to **give one term** in the sequence,
as well as the recurrence relation. You almost always give the **first value**, u_1.

Putting all of this together gives 5, 8, 11, 14, 17, ... as $u_{n+1} = u_n + 3,\ u_1 = 5$.

***Arithmetic Progressions Add** a **Fixed Amount** each time*

The **first term** of an arithmetic progression is given the symbol a. The **amount you add** each time is called the common difference, or d. The **position of any term** in the sequence is called n.

Term	n		
1st	1	a	
2nd	2	$(a) + d$	
3rd	3	$(a + d) + d$	$= a + 2d$
4th	4	$(a + 2d) + d$	$= a + 3d$
.	.	.	
.	.	.	
n^{th}	n	$a + (n - 1)d$	

Each term is made up of the previous one plus d. It's a sort of recurrence relation.

This is the formula for the n^{th} term.

Example: Find the 20th term of the arithmetic progression 2, 5, 8, 11, ... and find the formula for the n^{th} term.

Here $a = 2$ and $d = 3$. ← To get d, just find the difference between two terms next to each other — e.g. 11 − 8 = 3.

So 20th term $= a + (20 - 1)d$
$= 2 + 19 \times 3 = 59$

The **general term** is the **n^{th} term**, i.e. $a + (n - 1)d$
$= 2 + (n - 1)3$
$= 3n - 1$

Sequences

Geometric Progressions Multiply by a Constant each time

Geometric progressions work like this: the next term in the sequence is obtained by **multiplying the previous** one by a **constant value**. Couldn't be easier.

Ah, the classic chessboard example...

$u_1 = a \qquad\qquad\quad = a$ ← The first term (u_1) is called 'a'.

$u_2 = a \times r \qquad\quad = ar$ ← The number you multiply by each time is called 'the common ratio', symbolised by 'r'.

$u_3 = a \times r \times r \quad = ar^2$

$u_4 = a \times r \times r \times r = ar^3$

Here's the formula describing any term in the geometric progression: $u_n = ar^{n-1}$

Example: There is a 64-square chessboard with a 1p piece on the first square, 2p on the second square, 4p on the third, 8p on the fourth and so on. Calculate how much money is on the board.

This is a **geometric progression**, where you get the next term in the sequence by multiplying the previous one by 2.

So $a = 1$ (because you start with 1p on the first square) and $r = 2$.

So $u_1 = 1, u_2 = 2, u_3 = 4, u_4 = 8$ ...

To be continued... (once we've gone over how to sum the terms of a geometric progression on p.48)

Sequences can be Increasing, Decreasing or Periodic (or none of these)

In an **increasing sequence**, each term is larger than the previous term, so $u_{k+1} > u_k$ for all terms. The sequence of square numbers (1, 4, 9, 16, ...) is an increasing sequence.

In a **decreasing sequence**, each term is smaller than the previous term, so $u_{k+1} < u_k$ for all terms. The sequence 10 000, 1000, 100, 10, ... is a decreasing sequence.

In a **periodic sequence**, the terms **repeat** in a cycle. The number of terms in one cycle is known as the **order**. The sequence 1, 0, 1, 0, ... is a periodic sequence with order 2.

Some sequences are neither increasing, decreasing nor periodic — for example 1, –2, 3, –4, 5, ...

Example: A sequence has n^{th} term $7n - 16$. Show that the sequence is increasing.

Use the n^{th} term formula to find u_k and u_{k+1}: $u_k = 7k - 16$, $u_{k+1} = 7(k + 1) - 16 = 7k - 9$

Show that $u_{k+1} > u_k$ for all k: $7k - 9 > 7k - 16 \Rightarrow -9 > -16$

This is true so it is an **increasing sequence**.

Here, $u_{k+1} > u_k$ for any value of k, but you only need to show it's true for integer values of k.

Practice Questions

Q1 Describe the arithmetic sequence 32, 37, 42, 47, ... using a recurrence relation.

Q2 Find the common difference in an arithmetic sequence that starts with –2, ends with 19 and has 29 terms.

Q3 For the geometric progression 2, –6, 18, ... find: a) the common ratio, b) the 10^{th} term.

Q4 Show that the sequence with n^{th} term $55 - 3n$ is decreasing.

Exam Questions

Q1 Ned has 15 cuboid pots that need filling with soil. Each pot is taller than the one before it. The different capacities of his 15 pots form an arithmetic sequence with first term (representing the smallest pot) a ml and the common difference d ml. The 7^{th} pot has a capacity of 580 ml and largest pot has a capacity of 1020 ml. Find the value of a and the value of d. [5 marks]

Q2 A geometric sequence has first term 12 and common ratio 1.3. Find the value of the tenth term in the sequence. [2 marks]

Triangle, square, pentagon — that's my idea of a geometric progression...

Make sure you understand the difference between arithmetic and geometric progressions — you need to be really happy with them for the next couple of pages, so it's worth spending a bit of time getting your head around them now.

There's ***Another*** *way of* ***Writing Series****, too*

So far, the letter S has been used for the sum. The Greeks did a lot of work on this — their capital letter for S is Σ or **sigma**. This is used today, together with the general term, to mean the **sum** of the series.

A useful result is $\sum_{1}^{n} 1 = n$.

Example: Find $\sum_{n=1}^{15}(2n+3)$

...and ending with n = 15

Starting with n = 1...

This means you have to find the sum of the **first 15 terms** of the series with n^{th} term $2n + 3$.
The first term ($n = 1$) is 5, the second term ($n = 2$) is 7, the third is 9, ... and the last term ($n = 15$) is 33.
In other words, you need to find $5 + 7 + 9 + ... + 33$. This gives $a = 5$, $d = 2$, $n = 15$ and $l = 33$.
You know all of a, d, n and l, so you can use either formula:

It doesn't matter which method you use.

$$S_n = n \times \frac{(a+l)}{2} \qquad S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = 15 \times \frac{(5+33)}{2} = 15 \times 19 \qquad S_{15} = \frac{15}{2}[2 \times 5 + 14 \times 2] = \frac{15}{2}[10 + 28]$$

$$\mathbf{S_{15} = 285} \qquad \mathbf{S_{15} = 285}$$