# Arithmetic Series

*OK, now you know what a sequence is, it's time to move on to series. A series is very like a sequence, but there is one very important difference — in a series, you add the terms together.*

## A Series is when you Add the Terms to Find the Total

$S_n$ is the total of the first $n$ terms of the arithmetic progression:

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + (n - 1)d)$$

or: $$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$$

The l stands for the <u>last value</u> in the progression. You work it out as $l = a + (n - 1)d$

There's a really neat formula you can use to find $S_n$: $S_n = n \times \frac{(a+l)}{2}$

Here's the proof of the formula:

$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$ — Write $S_n$ in terms of a and l.

$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$ — Then write the same thing with the terms in reverse order.

$2S_n = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$ — Now, add these together, term by term. All the ds cancel and you're left with (a + l), n times.

$2S_n = n(a + l)$ — Now just divide by 2 to get the formula.

If you don't like formulas, just think of it as the **average** of the **first and last** terms multiplied by the **number of terms**.

**Example:** Find the sum of the arithmetic series with first term 3, last term 87 and common difference 4.

Use the information about the last value, $l$: $a + (n - 1)d = 87$

Then **plug in** the other values: $3 + 4(n - 1) = 87$ — Here you know a, d and l, but you don't know n yet.

$4n - 4 = 84$

$4n = 88 \Rightarrow n = 22$ — n = 22 means that there are 22 terms in the progression.

So $S_{22} = 22 \times \frac{(3+87)}{2} = \mathbf{990}$

The $S_n$ formula is on the formula sheet as $S_n = \frac{1}{2}n(a + l)$, which is equivalent to the formula above.

## They Won't always give you the Last Term

Don't panic though — there's a formula to use when the **last term is unknown**.

You know $l = a + (n - 1)d$ and $S_n = n \times \frac{(a+l)}{2}$. Plug $l$ into $S_n$ and rearrange to get this formula, which is also on the formula sheet:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

**Example:** For the arithmetic sequence –5, –2, 1, 4, 7, … find the sum of the first 20 terms.

So $a = -5$ and $d = 3$. The question says $n = 20$ too.

$S_{20} = \frac{20}{2}[2 \times (-5) + (20 - 1) \times 3]$

$= 10[-10 + 19 \times 3]$

$\mathbf{S_{20} = 470}$

## There's Another way of Writing Series, too

So far, the letter $S$ has been used for the sum. The Greeks did a lot of work on this — their capital letter for S is Σ or **sigma**. This is used today, together with the general term, to mean the **sum** of the series.

**Example:** Find $\sum_{n=1}^{15}(2n + 3)$ — Starting with n = 1... ...and ending with n = 15

A useful result is $\sum_{1}^{n} 1 = n$.

This means you have to find the sum of the **first 15 terms** of the series with $n^{th}$ term $2n + 3$.

The first term ($n = 1$) is **5**, the second term ($n = 2$) is **7**, the third is **9**, … and the last term ($n = 15$) is **33**.

In other words, you need to find 5 + 7 + 9 + … + 33. This gives $a = 5$, $d = 2$, $n = 15$ and $l = 33$.

You know all of $a$, $d$, $n$ and $l$, so you can use either formula:

It doesn't matter which method you use.

$S_n = n \times \frac{(a+l)}{2}$

$S_{15} = 15 \times \frac{(5+33)}{2} = 15 \times 19$

$\mathbf{S_{15} = 285}$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{15} = \frac{15}{2}[2 \times 5 + 14 \times 2] = \frac{15}{2}[10 + 28]$

$\mathbf{S_{15} = 285}$

# Arithmetic Series

## Use Arithmetic Progressions to add up the Natural Numbers

Natural numbers are just positive whole numbers.

The **sum of the first $n$ natural numbers** looks like this:

$$S_n = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

So $a = 1$, $l = n$ and also $n = n$.
Now just plug those values into the formula:

$$S_n = n \times \frac{(a+l)}{2} \Rightarrow S_n = \frac{1}{2}n(n+1)$$

It's pretty easy to **prove** this:

1) Say, $S_n = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$ ①
2) ① is just addition, so it's also true that:
$S_n = n + (n-1) + (n-2) + \dots + 3 + 2 + 1$ ②
3) Add ① and ② together to get:
$2S_n = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1)$
$\Rightarrow 2S_n = n(n+1) \Rightarrow S_n = \frac{1}{2}n(n+1)$. Voilà.

**Example:** On Day 1, Erica puts £1 in her piggy bank. On Day 2, she puts in £2, on Day 3 she puts in £3, etc. How much money will she have after 100 days?

This sounds pretty hard, but it's just asking for the sum of the whole numbers from 1 to 100 — so all you have to do is stick it into the formula:

$$S_{100} = \frac{1}{2} \times 100 \times 101 = \textbf{£5050}$$

**Example:** The sum of the first $k$ natural numbers is 861. Find the value of $k$.

Form an equation in $k$:
$$\frac{1}{2}k(k+1) = 861$$
$$k^2 + k = 1722$$
$$k^2 + k - 1722 = 0$$

Now factorise:
$$(k+42)(k-41) = 0$$
$$k = -42 \text{ or } k = 41$$

$k$ can't be negative so $k = 41$

## Practice Questions

Q1 Find the sum of the arithmetic series that starts with 7, ends with 35 and has 8 terms.

Q2 Find the sum of the arithmetic series that begins with 5, 8, ... and ends with 65.

Q3 An arithmetic series has first term 7 and fifth term 23.
Find: a) the common difference, b) the 15th term, c) the sum of the first 10 terms.

Q4 An arithmetic series has 7th term 36 and 10th term 30. Find the $n$th term and the sum of the first five terms.

Q5 Find: a) $\sum_{n=1}^{20}(3n-1)$ b) $\sum_{n=1}^{10}(48-5n)$

**Exam Questions**

Q1 An arithmetic sequence $a_1, a_2, a_3, \dots$ is defined by $a_1 = k$, $a_{n+1} = 3a_n + 11$, $n \geq 1$, where $k$ is a constant.
a) Show that $a_4 = 27k + 143$. [3 marks]
b) Find the value of $k$, given that $\sum_{r=1}^{4} a_r = 278$. [3 marks]

Q2 Ed's personal trainer has given him a timetable to improve his upper-body strength, which gradually increases the amount of push-ups Ed does by the same amount each day.

The timetable for the first four days is shown:

| Day: | Mon | Tue | Wed | Thur |
|---|---|---|---|---|
| Number of push-ups: | 6 | 14 | 22 | 30 |

a) Find an expression, in terms of $n$, for the number of push-ups he will have to do on day $n$. [2 marks]

b) Calculate how many push-ups Ed will have done in total if he follows his routine for 10 days. [1 mark]

The trainer recommends that Ed takes a break when he has done a cumulative total of 2450 push-ups.

c) Given that Ed completes his exercises on day $k$, but reaches the recommended limit part-way through day $(k+1)$, show that $k$ satisfies $(2k-49)(k+25) < 0$ and find the value of $k$. [5 marks]

## This sigma notation is all Greek to me...

*A sequence is just a list of numbers (with commas between them) and a series is when you add all the terms together. It doesn't sound like a big difference, but mathematicians get all hot under the collar when you get the two mixed up. Remember that BlackADDer was a great TV series, not a TV sequence. (Sounds daft, but I bet you remember it now.)*