Geometric Series

Remember that a geometric sequence is one where you multiply by a common ratio to get from one term to the next. If you add the terms in a geometric sequence, you get a geometric series.

*There's a **Formula** for the **Sum** of a **Geometric Series***

To work out the formula for the sum of a geometric progression, you use **two series** and **subtract**:

For a geometric progression: $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

Multiplying by r gives: $rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-2}\ ar^{n-1} + ar^n$

Subtracting gives: $S_n - rS_n = a - ar^n$

Factorising: $(1 - r)S_n = a(1 - r^n)$ ⇒ $S_n = \frac{a(1-r^n)}{1-r}$

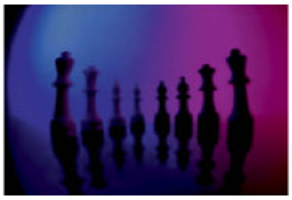
If the series were subtracted the other way around you'd get $S_n = \frac{a(r^n-1)}{r-1}$. Both versions are correct, but the (1 – r) version is the one that appears on the formula sheet.

Example: Calculate how much money is on the chessboard on p.45.

Here, you have to work out S_{64} (because there are 64 squares on a chessboard).
$a = 1$, $r = 2$ and $n = 64$. Putting these values into the formula gives:

$$S_{64} = \frac{1(1-2^{64})}{1-2} = \mathbf{1.84 \times 10^{19}} \text{ pence or } \mathbf{£1.84 \times 10^{17}}$$

≈ 16 million times the number of 1p coins in circulation



The chessboard had suddenly turned a bit sinister.

*Geometric progressions can either **Grow** or **Shrink***

In the chessboard example, each term was **bigger** than the previous one: 1, 2, 4, 8, 16, ...
You can create a series where each term is **smaller** than the previous one by using a **small value of *r***.

Example: If $a = 20$ and $r = \frac{1}{5}$, find the first five terms of the geometric sequence and the 20th term.

$u_1 = \mathbf{20}$, $u_2 = 20 \times \frac{1}{5} = \mathbf{4}$, $u_3 = 4 \times \frac{1}{5} = \mathbf{0.8}$, $u_4 = 0.8 \times \frac{1}{5} = \mathbf{0.16}$, $u_5 = 0.16 \times \frac{1}{5} = \mathbf{0.032}$

Each term is the previous one multiplied by r.

$u_{20} = 20 \times \left(\frac{1}{5}\right)^{19} = \mathbf{1.048576 \times 10^{-12}}$ ← The sequence is **tending towards zero**, but won't ever get there.

In general, for each term to be **smaller** than the one before, you need $|r| < 1$.
A sequence with $|r| < 1$ is called **convergent**, since the terms converge to a limit.
Any other sequence (like the chessboard example above) is called **divergent**.

|r| means the modulus (or size) of r, ignoring the sign of the number (see p.28). So |r| < 1 means that –1 < r < 1.

*A **Convergent Series** has a **Sum** to **Infinity***

In other words, if you just **kept** adding terms to a **convergent series**, you'd get **closer and closer** to a certain number, but you'd never actually reach it.

S_∞ just means 'sum to infinity'.

If $|r| < 1$ and n is very, very **big**, then r^n will be very, very **small** — or to put it technically, $r^n \to 0$ (try working out $(½)^{100}$ on your calculator if you don't believe me).

This means $(1 - r^n)$ is really, really close to 1.

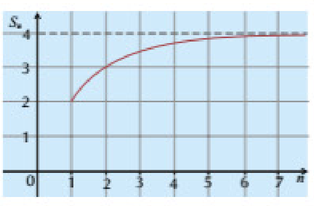
So, for $|r| < 1$, as $n \to \infty$, $S_n \to \frac{a}{1-r}$

This is given on the formula sheet as $S_\infty = \frac{a}{1-r}$

Example: If $a = 2$ and $r = \frac{1}{2}$, find the sum to infinity of the geometric series.

Using the **sum to infinity formula**: $S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = \mathbf{4}$

You can show this **graphically**:
The curve is getting **closer and closer** to 4, but it'll never actually get there.



You can see it getting closer and closer to 4 if you work out the individual sums:

$S_1 = 2$, $S_2 = 3$, $S_3 = 3\frac{1}{2}$, $S_4 = 3\frac{3}{4}$, $S_5 = 3\frac{7}{8}$, $S_6 = 3\frac{15}{16}$, $S_7 = 3\frac{31}{32}$ ...and so on... ...and so forth...

Geometric Series

A *Divergent* series *Doesn't* have a sum to infinity

Example: If $a = 2$ and $r = 2$, find the sum to infinity of the geometric series.

$u_1 = 2$ so $S_1 = \mathbf{2}$

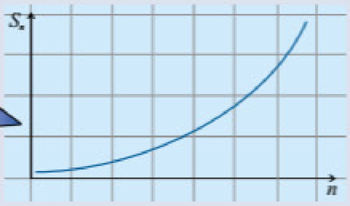
$u_2 = 2 \times 2 = 4$ so $S_2 = 2 + 4 = \mathbf{6}$

$u_3 = 4 \times 2 = 8$ so $S_3 = 2 + 4 + 8 = \mathbf{14}$

$u_4 = 8 \times 2 = 16$ so $S_4 = 2 + 4 + 8 + 16 = \mathbf{30}$

$u_5 = 16 \times 2 = 32$ so $S_5 = 2 + 4 + 8 + 16 + 32 = \mathbf{62}$

This is an exponential graph — see p.77.



As $n \to \infty$, $S_n \to \infty$ in a big way. So big, in fact, that eventually you **can't work it out** — so don't bother. There is **no sum to infinity** for a **divergent** series.

Example: On a child's 1st birthday, £3000 is invested in an account with a fixed compound interest rate of 4% per year. The interest is paid in every year, on the child's birthday.

a) What will the account be worth on the child's 7th birthday?

b) When will the account have doubled in value?

It's compound interest, so multiply by 1.04 each year.

a) $u_1 = a = \mathbf{3000}$

$u_2 = 3000 + (4\% \text{ of } 3000)$ ← This is the interest.

$= 3000 + (0.04 \times 3000) = 3000(1 + 0.04)$

$= \mathbf{3000 \times 1.04}$ ← So r = 1.04

$u_3 = u_2 \times 1.04 \times 2 = (3000 \times 1.04) \times 1.04$

$= \mathbf{3000 \times (1.04)^2}$

$u_4 = \mathbf{3000 \times (1.04)^3}$

$\vdots \quad \vdots \quad \vdots$

$u_7 = 3000 \times (1.04)^6$

$=$ **£3795.96** (to the nearest penny)

b) You need to know when $u_n > 6000$ (double the original value).

From part a) you know that $u_n = 3000 \times (1.04)^{n-1}$

So $3000 \times (1.04)^{n-1} > 6000$

$(1.04)^{n-1} > 2$

To complete this you need to use **logs** (see p.76):

$\log(1.04)^{n-1} > \log 2$

$(n-1)\log(1.04) > \log 2$

$n - 1 > \dfrac{\log 2}{\log 1.04}$

$n - 1 > 17.67$

$n > \mathbf{18.67}$ (2 d.p.)

So u_{19} (the amount at the start of the 19th year) will be more than double the original amount.

So the account will have doubled in value when the interest is paid in on the child's **19th** birthday.

Practice Questions

Q1 Find the sum of the first 12 terms of the following geometric series:

a) $2 + 8 + 32 + \ldots$ b) $30 + 15 + 7.5 + \ldots.$

Q2 Find the common ratio for the following geometric series.
State which ones are convergent and which are divergent.

a) $1 + 2 + 4 + \ldots$ b) $81 + 27 + 9 + \ldots$ c) $1 + \frac{1}{3} + \frac{1}{9} + \ldots$

Q3 For the geometric progression 24, 12, 6, ..., find:

a) the common ratio, b) the seventh term,

c) the sum of the first 10 terms, d) the sum to infinity.

Exam Question

Q1 A geometric series has second term $u_2 = 5$ and sum to infinity $S_\infty = 36$.

a) Show that $36r^2 - 36r + 5 = 0$, where r represents the two possible values of the common ratio. [3 marks]

b) Hence find the possible values of r, and the two corresponding first terms. [4 marks]

To infinity and beyond (unless it's a convergent series, in which case, to 4)...

I find it odd that I can keep adding things to a sum forever, but the sum never gets really really big. If this is really blowing your mind, draw a quick sketch of the graph (like in the example above) so you can see what's happening.