Angles, Arc Length and Sector Area

You should be familiar with angles measured in degrees. For A Level Maths, you also need to measure them in radians.

Radians** are another way of **Measuring Angles

You need to know how radians relate to **degrees**.
In short, **180 degrees = π radians**. The table below shows you how to convert between the two units:

Converting angles	
Radians to degrees: Divide by π, multiply by 180.	Degrees to radians: Divide by 180, multiply by π.

Here's a table of some of the **common angles** you're going to need — in degrees and radians:

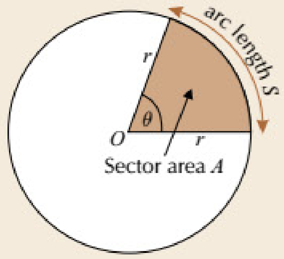
Degrees	0	30	45	60	90	120	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$	2π

"...and bake in an oven preheated to π radians for around 40 minutes."

Angles, Arc Length and Sector Area

*Find the **Length of an Arc** and the **Area of a Sector** of a circle*

If you have a **sector of a circle** (like a section of pie chart), you can work out the **length** of the **curved side** (the **arc**) and its **area** — as long as you know the **angle** at the centre ($\boldsymbol{\theta}$, in **radians**) and the length of the **radius** (r).



For a circle with a **radius of r**, where the angle θ is measured in **radians**, you can work out S, the **length of the arc**, using:

$$S = r\theta$$

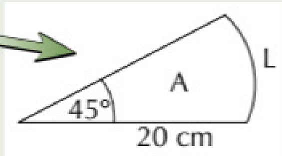
To find A, the **area of the sector**, use:

$$A = \frac{1}{2}r^2\theta$$

If you put $\theta = 2\pi$ into either formula (and so make the sector equal to the whole circle), you get the normal circumference and area formulas.

*Use **Radians** with **Both Formulas***

Example: Find the exact length L and area A in the diagram.



Right, first things first... it's an **arc length** and **sector area**, so you need the angle in **radians**.

$45° = \frac{45 \times \pi}{180} = \frac{\pi}{4}$ radians

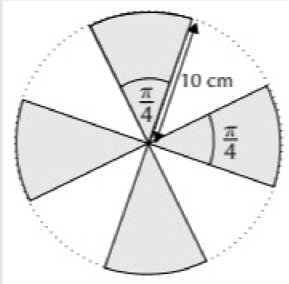
Or you could just quote this if you've learnt the stuff on the previous page.

Now bung everything in your formulas:

$L = r\theta = 20 \times \frac{\pi}{4} = \mathbf{5\pi\ cm}$

$A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 20^2 \times \frac{\pi}{4} = \mathbf{50\pi\ cm^2}$

Example: Find the area of the shaded part of the symbol.



You need the area of the 'leaves' and so use the formula $\frac{1}{2}r^2\theta$.

Each leaf has area $\frac{1}{2} \times 10^2 \times \frac{\pi}{4} = \frac{25\pi}{2}$ cm²

So the area of the whole symbol $= 4 \times \frac{25\pi}{2} = \mathbf{50\pi\ cm^2}$

You could also use the total angle of all the shaded sectors (π) to go straight to the answer. Or you could notice that exactly half of the area of the circle is shaded.

Practice Questions

Q1 Write down the exact value of: a) $\frac{3\pi}{2}$ radians in degrees, b) 120° in radians.

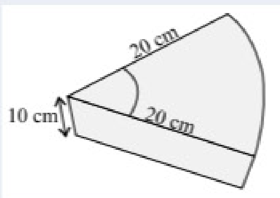
Q2 Calculate the value of the angle 50° in radians, to 3 decimal places.

Q3 Write down the exact value of: a) cos 30°, b) sin 45°, c) tan 60°, d) sin $\frac{\pi}{6}$.

Q4 A sector of a circle with radius 5 cm has an arc length of 15 cm. Find the centre angle, in degrees to 1 d.p.

Exam Question

Q1 The diagram shows the dimensions of a child's wooden toy.
The toy has a constant cross-section and a height of 10 cm.
Its cross-section is a sector of a circle with radius 20 cm and angle $\frac{\pi}{4}$ radians.



a) Show that the volume of the toy, $V = 500\pi$ cm³. [3 marks]

b) Show that the surface area of the toy, $S = (150\pi + 400)$ cm². [4 marks]

π = 3.14159265358979323846264338327950288419716939 9... (make sure you know it)

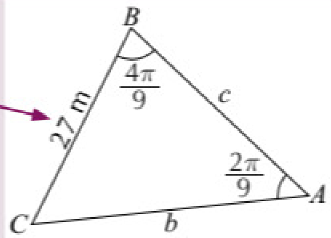
It's worth repeating, just to make sure — the formulas for arc length and sector area only work with angles in radians.

Trig Formulas and Identities

*The **Formulas** work with **Radians** as well as **Degrees***

Example: In the triangle ABC, $A = \frac{2\pi}{9}$ radians, $a = 27$ m and $B = \frac{4\pi}{9}$ radians. Find the missing angles and sides, and calculate the area of the triangle.

1) Draw a quick sketch first — don't worry if it's not deadly accurate, though.



Make sure you put side a opposite angle A.

2) You're given 2 angles and a side, so you need the **sine rule**.

The angles in a triangle add up to π radians.

First of all, get the other angle: $\angle C = \pi - \frac{2\pi}{9} - \frac{4\pi}{9} = \mathbf{\frac{\pi}{3}}$ **radians**

3) Then find the other sides, one at a time:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{27}{\sin\frac{2\pi}{9}} = \frac{b}{\sin\frac{4\pi}{9}}$$

$$\Rightarrow b = \frac{\sin\frac{4\pi}{9}}{\sin\frac{2\pi}{9}} \times 27 = 41.366... = \mathbf{41.4\ m}\ \text{(1 d.p.)}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin\frac{\pi}{3}} = \frac{27}{\sin\frac{2\pi}{9}}$$

$$\Rightarrow c = \frac{\sin\frac{\pi}{3}}{\sin\frac{2\pi}{9}} \times 27 = \mathbf{36.4\ m}\ \text{(1 d.p.)}$$

4) Now just use the formula to find its area:

$$\text{Area } \Delta ABC = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2} \times 27 \times 41.366... \times \sin\frac{\pi}{3}$$
$$= \mathbf{483.6\ m^2}\ \text{(1 d.p.)}$$

Use a more accurate value for b here, rather than the rounded value 41.4.

*Use the **Formulas** to find the **Exact Value** of trig expressions*

You should know the value of sin, cos and tan for **common angles** (in **degrees** and **radians**). These values come from using **Pythagoras** on **right-angled triangles** — see p.56.

In the exam you might be asked to calculate the **exact value** of sin, cos or tan for **another angle** using your knowledge of those angles and the **addition formulas**.

Find a **pair of angles** from the table which **add or subtract** to give the angle you're after. Then plug them into the **addition formula**, and work it through.

	0°	30°	45°	60°	90°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	n/a

You have to know the graphs in **Radians** too

The graphs of sin x, cos x and tan x are exactly the **same shape** whether x is in **radians** or **degrees** — you just need to remember the **key points** of the graphs in both units.

$x°$	–360	–270	–180	–90	0	90	180	270	360
x radians	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin x	0	1	0	–1	0	1	0	–1	0
cos x	1	0	–1	0	1	0	–1	0	1
tan x	0	—	0	—	0	—	0	—	0

The '—' here means tan x is undefined, so draw an asymptote.

Further Trig

Just when you thought you'd seen all the functions that trigonometry could throw at you, here come six more. The ones at the bottom of the page are particularly handy when you're solving trig equations.

Arcsin, Arccos and Arctan are the Inverses of Sin, Cos and Tan

In Section 2 you saw that some functions have **inverses**, which reverse the effect of the function. The **trig functions** have inverses too.

ARCSINE is the **inverse of sine**. You might see it written as **arcsin** or $\sin^{-1}$.
ARCCOSINE is the **inverse of cosine**. You might see it written as **arccos** or $\cos^{-1}$.
ARCTANGENT is the **inverse of tangent**. You might see it written as **arctan** or $\tan^{-1}$.

You should have buttons for doing arcsin, arccos and arctan on your calculator — they'll probably be labelled $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

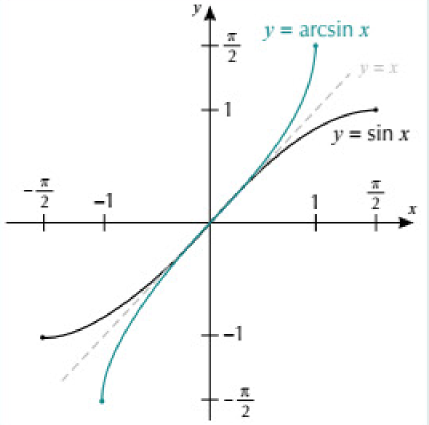
The inverse trig functions **reverse** the effect of sin, cos and tan. For example, if $\sin 30° = 0.5$, then $\arcsin 0.5 = 30°$.

To Graph the Inverse Functions you need to Restrict their Domains

1) The functions sine, cosine and tangent are NOT **one-to-one mappings** (see p.34) — lots of values of x give the same value for $\sin x$, $\cos x$ or $\tan x$. For example: $\cos 0 = \cos 2\pi = \cos 4\pi = 1$, and $\tan 0 = \tan \pi = \tan 2\pi = 0$.
2) Only **one-to-one functions** have inverses, so for the inverse to be a function you have to **restrict the domain** of the trig function to make it one-to-one (see graphs below). This means that you only plot the graphs between certain x values, so that for **each x value**, you end up with **one y value**.
3) As the graphs are inverse functions, they're also **reflections** of the sin, cos and tan functions in the line $y = x$.

Arcsine

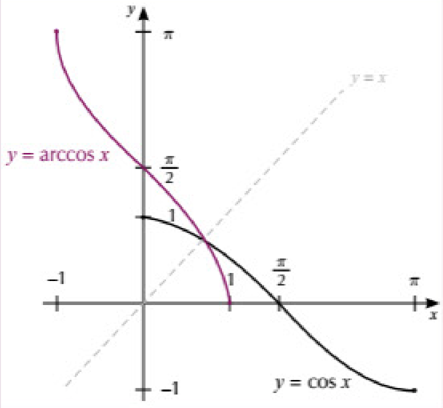
For arcsin, limit the domain of $\sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (the range of $\sin x$ is still $-1 \le \sin x \le 1$). This means the domain of $\arcsin x$ is $-1 \le x \le 1$ and its range is $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$.



This graph goes through the origin.
The coordinates of its endpoints are $\left(1, \frac{\pi}{2}\right)$ and $\left(-1, -\frac{\pi}{2}\right)$.

Arccosine

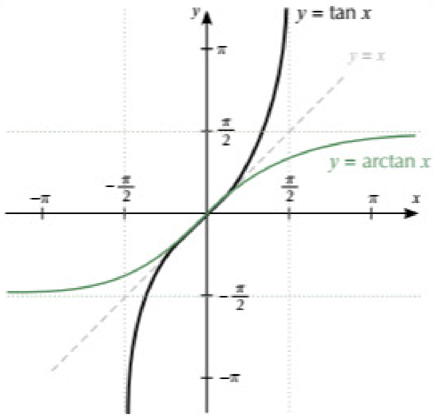
For arccos, limit the domain of $\cos x$ to $0 \le x \le \pi$ (the range of $\cos x$ is still $-1 \le \cos x \le 1$).
This means the domain of $\arccos x$ is $-1 \le x \le 1$ and its range is $0 \le \arccos x \le \pi$.



This graph crosses the y-axis at $\left(0, \frac{\pi}{2}\right)$. The coordinates of its endpoints are $(-1, \pi)$ and $(1, 0)$.

Arctangent

For arctan, limit the domain of $\tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (this doesn't limit the range of $\tan x$).
This means that the domain of $\arctan x$ isn't limited, but its range is limited to $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$.



This graph goes through the origin.
It has asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

Cosec, Sec and Cot are the Reciprocals of Sin, Cos and Tan

When you take the **reciprocal** of the three main trig functions, sin, cos and tan, you get three new trig functions — **cosecant** (or **cosec**), **secant** (or **sec**) and **cotangent** (or **cot**).

$$\operatorname{cosec} \theta \equiv \frac{1}{\sin \theta} \qquad \sec \theta \equiv \frac{1}{\cos \theta} \qquad \cot \theta \equiv \frac{1}{\tan \theta}$$

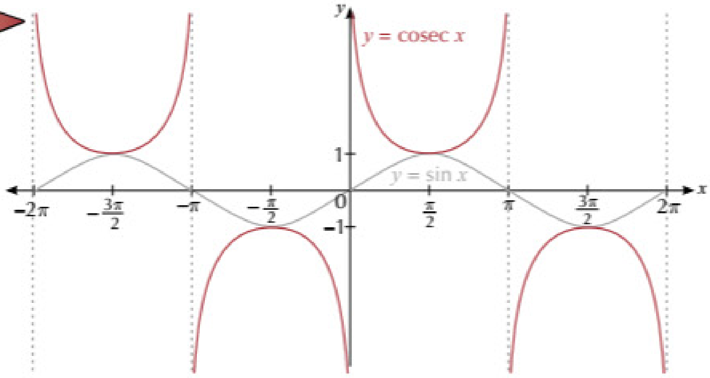
The trick for remembering which is which is to look at the third letter — co**s**ec (1/**s**in), se**c** (1/**c**os) and co**t** (1/**t**an).

Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, you can also think of **cot θ** as being $\frac{\cos \theta}{\sin \theta}$.

Further Trig

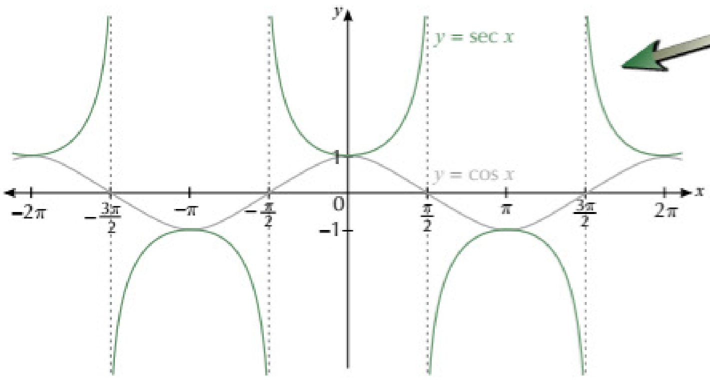
Graphing Cosec, Sec and Cot

Cosec This is the graph of $y = \text{cosec } x$.



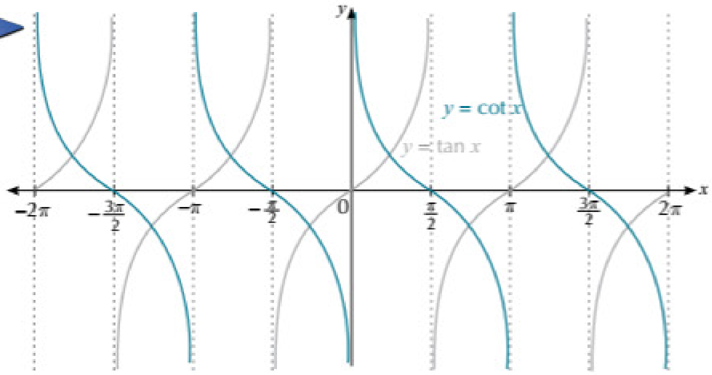
1) Since $\text{cosec } x = \frac{1}{\sin x}$, $y = \text{cosec } x$ is **undefined** at any point where $\sin x = 0$. So cosec x has **asymptotes** at $x = n\pi$ (where n is any integer).
2) The graph of cosec x has **minimum** points at $y = 1$ (wherever the graph of sin x has a maximum).
3) It has **maximum** points at $y = -1$ (wherever sin x has a minimum).

Sec This is the graph of $y = \sec x$.



1) As $\sec x = \frac{1}{\cos x}$, $y = \sec x$ is **undefined** at any point where $\cos x = 0$. So sec x has **asymptotes** at $x = \left(n\pi + \frac{\pi}{2}\right)$ (where n is any integer).
2) The graph of sec x has **minimum** points at $y = 1$ (wherever the graph of cos x has a maximum).
3) It has **maximum** points at $y = -1$ (wherever cos x has a minimum).

Cot This is the graph of $y = \cot x$.



1) Since $\cot x = \frac{1}{\tan x}$, $y = \cot x$ is **undefined** at any point where $\tan x = 0$. So cot x has **asymptotes** at $x = n\pi$ (where n is any integer).
2) $y = \cot x$ **crosses the x-axis** at every place where the graph of tan x has an **asymptote** — this is any point with the coordinates $\left(\left(n\pi + \frac{\pi}{2}\right), 0\right)$.

Practice Questions

Q1 Giving your answers in radians, find the exact values of: a) arcsin 1, b) $\arccos \frac{1}{\sqrt{2}}$, c) $\arctan \sqrt{3}$.

Q2 For $\theta = 30°$, find the exact values of: a) cosec θ, b) sec θ, c) cot θ.

Q3 Solve $\text{cosec } x = \sqrt{2}$, where $0° \le x \le 360°$.

Exam Questions

Q1 Solve, to 3 significant figures, the equation $\arccos x = 2$ radians for the interval $0 \le \arccos x \le \pi$. [2 marks]

Q2 a) Sketch the graph of $y = \text{cosec } x$ for $-\pi \le x \le \pi$. [3 marks]

b) Solve the equation $\text{cosec } x = \frac{5}{4}$ for $-\pi \le x \le \pi$. Give your answers correct to 3 significant figures. [3 marks]

c) Solve $\text{cosec } x = 3 \sec x$ for $-\pi \le x \le \pi$. Give your answers correct to 3 significant figures. [3 marks]

Why did I multiply cot x by sin x? Just 'cos...

If I were you, I'd get the shapes of all these graphs memorised — you can find the range and domain of the functions just by looking at them (and they're pretty, too). You might also have to transform a trig graph — use the same method as you would for other graphs (see p.31). And remember that tip about looking at the third letter — it's a belter.