Further Trig Identities and Approximations

The *Small Angle Approximations* simplify equations too

When an angle θ (measured in **radians**) is **very small** (**< 1**), you can approximate the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$ using the **small angle approximations**:

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2$$

Example: Approximate cos 10° to 6 d.p, and find the percentage error.

1) First convert the angle to radians, and check it's small enough:

$10° = \frac{10\times\pi}{180} = 0.1746329...$ radians

θ is a lot smaller than 1, so you can use the approximations.

2) The small angle approximation for cos is $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, so:

$$\cos 0.1746329... \approx 1 - \frac{1}{2}(0.1746329...)^2 = 0.984769 \text{ (6 d.p.)}$$

3) The actual value of cos 10° = 0.984808 (6 d.p.),

so the percentage error in the approximation $= \frac{\text{actual value} - \text{approximation}}{\text{actual value}} \times 100\%$

$$= \frac{0.984808 - 0.984769}{0.984808} \times 100\% = 0.004\% \text{ (3 d.p.)}$$

So the small angle approximation is pretty accurate.

You can use them to approximate more **complicated functions**, involving sin, cos and tan of **multiples** of θ (i.e. $n\theta$ when $n\theta < 1$).

Make sure that you apply the approximation to **everything** inside the trig function, e.g. $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, $\cos 3\theta \approx 1 - \frac{1}{2}(3\theta)^2$.

Example: Find an approximation for $f(x) = 4\cos 2x \tan 3x$ when x is small.

Replace cos and tan with the small angle approximations:

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ *$\tan\theta \approx \theta$*

$$f(x) \approx 4 \times (1 - \tfrac{1}{2}(2x)^2) \times (3x)$$
$$= (4 - 8x^2) \times 3x$$
$$= 12x - 24x^3 \text{ or } 12x(1 - 2x^2)$$

As with the identities, you'll be expected to use the small angle approximations in **proofs** and '**show that**' questions — see pages 74-75.

Practice Questions

Q1 Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to produce the identity $\sec^2\theta \equiv 1 + \tan^2\theta$.

Q2 Use the identity $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$ to simplify and solve $\text{cosec}^2\theta = -2\cot\theta$, for $0 \le \theta \le \pi$.

Q3 Use the small angle approximations for: a) sin 0.256, b) cos 0.02, c) 2 tan 0.1 sin 0.1.

Exam Questions

Q1 a) (i) Show that $3\tan^2\theta - 2\sec\theta = 5$ can be written as $3\sec^2\theta - 2\sec\theta - 8 = 0$. [2 marks]

(ii) Hence or otherwise show that $\cos\theta = -\frac{3}{4}$ or $\cos\theta = \frac{1}{2}$. [3 marks]

b) Use your results from part a) to solve the equation $3\tan^2 2x - 2\sec 2x = 5$ for $0 \le x \le 180°$, to 2 d.p. [3 marks]

Q2 a) Find an approximation for $\theta\sin\left(\frac{\theta}{2}\right) - \cos\theta$, when θ is small. [2 marks]

b) Hence give an approximate value of $\theta\sin\left(\frac{\theta}{2}\right) - \cos\theta$ when $\theta = 0.1$. [1 mark]

This section has more identities than Clark Kent...

I was devastated when my secret identity was revealed — I'd been masquerading as a mysterious caped criminal mastermind with an army of minions and a hidden underground lair. It was great fun, but I had to give it all up and write about trig. And lucky for you I did — who else would distract you from solving equations with cute bunny pics?