Further Trig Identities and Approximations

Ahh, more trig identities. More useful than a monkey wrench, and more fun than a pair of skateboarding rabbits. Or have I got that the wrong way round...

*Learn these **Three Trig Identities***

Hopefully you're familiar with this handy little **trig identity** by now:

Remember, the $\equiv$ sign tells you that this is true for all values of θ, rather than just certain values.

Identity 1: $$\cos^2\theta + \sin^2\theta \equiv 1$$

You can use this one to produce a couple of other identities that you need to know about...

Identity 2: $$\sec^2\theta \equiv 1 + \tan^2\theta$$

To get this, you just take everything in Identity 1, and **divide** it by $\cos^2\theta$:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

Remember that $\cos^2\theta = (\cos\theta)^2$.

Identity 3: $$\text{cosec}^2\theta \equiv 1 + \cot^2\theta$$

You get this one by **dividing** everything in Identity 1 by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 \equiv \text{cosec}^2\theta$$

*Use the **Trig Identities** to **Simplify Equations***

You can use any of the identities you've learnt to get rid of any trig functions that are making an equation difficult to solve.

More fun.

Example: Solve the equation $\cot^2 x + 5 = 4\,\text{cosec}\,x$ in the interval $0° \le x \le 360°$.

1) You can't solve this while it has **both** cot and cosec in it, so use **Identity 3** to swap $\cot^2 x$ for $\text{cosec}^2 x - 1$:

   $\text{cosec}^2 x - 1 + 5 = 4\,\text{cosec}\,x$

2) Now rearranging the equation gives:

   $\text{cosec}^2 x + 4 = 4\,\text{cosec}\,x \Rightarrow \text{cosec}^2 x - 4\,\text{cosec}\,x + 4 = 0$

3) So you've got a **quadratic** in cosec x — factorise it like you would any other quadratic equation.

   $\text{cosec}^2 x - 4\,\text{cosec}\,x + 4 = 0$ — If it helps, think of this as $y^2 - 4y + 4 = 0$.

   $(\text{cosec}\,x - 2)(\text{cosec}\,x - 2) = 0$ — Factorise it, and then replace the y with cosec x.

4) One of the brackets must be **equal to zero** — here they're both the same, so you only get **one equation**:

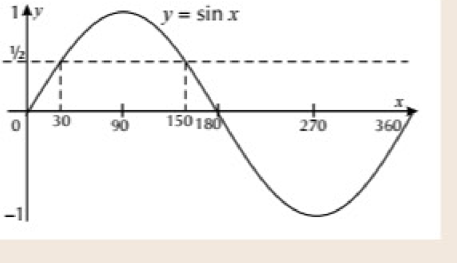
   $(\text{cosec}\,x - 2) = 0 \Rightarrow \text{cosec}\,x = 2$

5) Now you can convert this into $\sin x$, and solve it easily:

   $\text{cosec}\,x = 2 \Rightarrow \sin x = \frac{1}{2}$

   $x = 30°$ and $x = 150°$

To find the other values of x, draw a quick sketch of the sin curve. From the graph, you can see that sin x takes the value of $\frac{1}{2}$ twice in the given interval, once at x = 30° and once at x = 180 − 30 = 150°.



You could also use the CAST diagram (see p.62) — sin is positive in the 1st and 2nd quadrants, where x = 30° and 180° − 30° = 150°.

You'll also have to use these identities for trigonometric proofs, or to 'show that' one expression is the same as another. Look at pages 74-75 for some examples.

More useful.

Solving Trig Equations

I used to really hate trig stuff like this. But once I'd got the hang of it, I just couldn't get enough. I stopped going out, lost interest in romance — the CAST method became my life. Learn it, but be careful. It's addictive.

*There are **Two Ways** to find **Solutions** in an **Interval***

Example: Solve $\cos x = \frac{1}{2}$ for $-360° \le x \le 720°$.

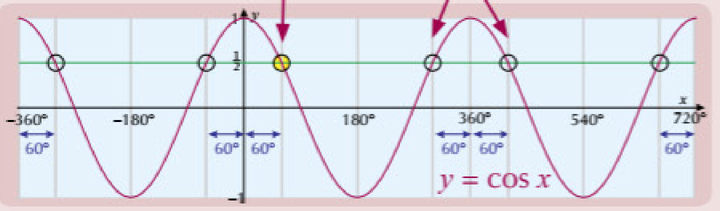
Like I said — there are **two ways** to solve this kind of question. Just use the one you prefer...

*You can draw a **graph**...*

1) Draw the **graph** of $y = \cos x$ for the range you're interested in...
2) Get the first solution from your **calculator** and mark this on the graph,
3) Use the **symmetry of the graph** to work out what the other solutions are:

Your calculator gives you a solution of 60°...

...then the other solutions are 60° either side of the graph's peaks.



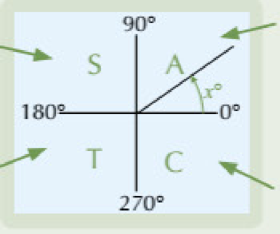
So the solutions are: **–300°**, **–60°**, **60°**, **300°**, **420°** and **660°**.

*...or you can use the **CAST** diagram*

CAST stands for **COS**, **ALL**, **SIN**, **TAN** — and the CAST diagram shows you where these functions are **positive**:

Between 90° and 180°, only <u>Sin</u> is positive.

Between 0 and 90°, <u>All</u> of sin, cos and tan are positive.



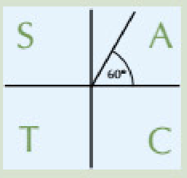
Between 180° and 270°, only <u>Tan</u> is positive.

Between 270° and 360°, only <u>Cos</u> is positive.

First, to find all the values of x between 0° and 360° where $\cos x = \frac{1}{2}$ — you do this:

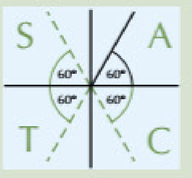
This is positive — so you're only interested in where cos is positive.

Put the **first solution** onto the **CAST diagram**.



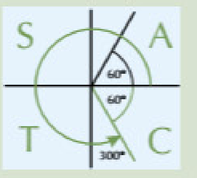
The angle from your calculator goes anticlockwise from the x-axis (unless it's negative — then it would go clockwise into the 4th quadrant).

Find the **other angles** between 0° and 360° that might be solutions.



The other possible solutions come from making the same angle from the horizontal axis in the other 3 quadrants.

Ditch the ones that are the **wrong sign**.



cos x = ½, which is positive. The CAST diagram tells you cos is positive in the 4th quadrant — but not the 2nd or 3rd — so ditch those two angles.

So you've got solutions 60° and 300° in the range 0° to 360°. But you need **all the solutions** in the range **–360° to 720°**. Get these by repeatedly **adding or subtracting 360°** onto each until you go out of range:

$x = 60° \Rightarrow$ (adding 360°) $x = 420°$, 780° (too big)

and (subtracting 360°) $x = -300°$, –660° (too small)

$x = 300° \Rightarrow$ (adding 360°) $x = 660°$, 1020° (too big)

and (subtracting 360°) $x = -60°$, –420° (too small)

So the solutions are: $x =$ **–300°**, **–60°**, **60°**, **300°**, **420°** and **660°**.

Solving Trig Equations

*Sometimes you end up with **sin kx = number**...*

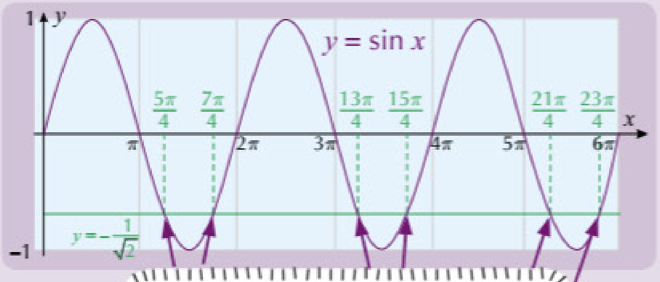
For these, it's definitely easier to draw the **graph** rather than use the CAST method — that's one reason why being able to sketch trig graphs properly is so important.

Example: Solve: $\sin 3x = -\frac{1}{\sqrt{2}}$ for $0 \le x \le 2\pi$.

1) You've got $3x$ instead of x, which means the interval you need to find solutions in is $\mathbf{0 \le 3x \le 6\pi}$. So draw the graph of $y = \sin x$ between 0 and 6π.

2) You should recall that $\frac{1}{\sqrt{2}}$ is $\sin\frac{\pi}{4}$ (see p.56), and so $-\frac{1}{\sqrt{2}}$ is $\sin\left(-\frac{\pi}{4}\right)$, which gives $\mathbf{3x = -\frac{\pi}{4}}$ — but this is **outside the interval** for $3x$, so use the pattern of the graph to find a solution in the interval.

   As the sin curve repeats every 2π, there'll be a solution at $2\pi - \frac{\pi}{4} = \mathbf{\frac{7\pi}{4}}$.



These are the solutions — but remember that this is for 3x. You'll need to divide by 3 to get your final answers.

3) Now use your graph to find the other 5 solutions. You can see that there's another solution at $\pi + \frac{\pi}{4} = \mathbf{\frac{5\pi}{4}}$.

4) Then add on 2π and 4π to both $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ to get: $3x = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}$ and $\frac{23\pi}{4}$

5) **Divide by 3** to get the solutions for x: $x = \mathbf{\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}}$ and $\mathbf{\frac{23\pi}{12}}$

6) **Check** your answers by putting these values into your calculator.

It really is mega-important that you check these answers — it's dead easy to make a silly mistake. They should all be in the range $0 \le x \le 2\pi$.

*...or **Something More Complicated...***

All the steps in this example are basically the same as in the one above, although at first sight it looks nightmarish. Just take it step by step and enjoy the modellinginess* of it all...

Example: A simplified model of the phases of the moon is given by $P = 50\sin\left(\left(\frac{90t}{7}\right)^\circ + 90^\circ\right) + 50$, where P is the percentage of the moon visible at night, and t is the number of days after a full (100%) moon was recorded. On which days, over 12 weeks, will there be a half moon?

1) Start by figuring out the interval for the solutions. 12 weeks = 84 days, so $0 \le t \le 84$.
   But you've got $\left(\left(\frac{90t}{7}\right)^\circ + 90^\circ\right)$, so the interval is:

$$\left(\frac{90\times 0}{7}\right)^\circ + 90^\circ \le \left(\left(\frac{90t}{7}\right)^\circ + 90^\circ\right) \le \left(\frac{90\times 84}{7}\right)^\circ + 90^\circ \Rightarrow \mathbf{90^\circ \le \left(\left(\frac{90t}{7}\right)^\circ + 90^\circ\right) \le 1170^\circ}$$

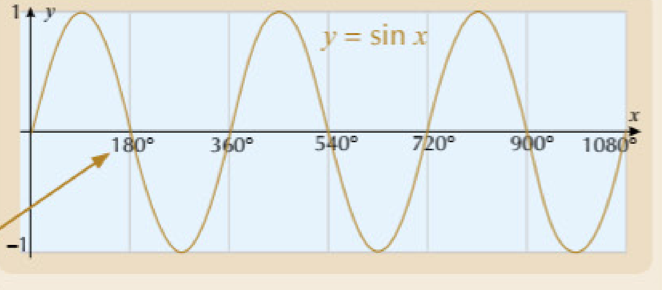
2) The question is asking you to find the values of t when $P = 50$, so put this into the model:

$$50 = 50\sin\left(\left(\frac{90t}{7}\right)^\circ + 90^\circ\right) + 50$$

$$\Rightarrow \quad 0 = \sin\left(\left(\frac{90t}{7}\right)^\circ + 90^\circ\right)$$

$$\Rightarrow \quad \left(\frac{90t}{7}\right)^\circ + 90^\circ = \sin^{-1} 0 = \mathbf{0^\circ}$$

   This is **outside the interval** for solutions, so consult the graph of $y = \sin x$ again.



3) You should know the x-intercepts of this graph off by heart by now — they're every 180° from zero.
   So $\left(\frac{90t}{7}\right)^\circ + 90^\circ = 180^\circ, 360^\circ, 540^\circ, 720^\circ, 900^\circ, 1080^\circ, 1260^\circ...$ ← This is above 1170° so it's not in the interval.

4) To solve for t, subtract 90° from each answer and divide by $\frac{90}{7}$: $t = \mathbf{7}, \mathbf{21}, \mathbf{35}, \mathbf{49}, \mathbf{63}$ and $\mathbf{77}$.
   So there will be a half moon **7**, **21**, **35**, **49**, **63** and **77** days after the first recorded full moon.

*Almost certainly a proper word somewhere.

Solving Trig Equations

*For equations with **tan x** in, it often helps to use this...*

$$\tan x \equiv \frac{\sin x}{\cos x}$$

You'll also have to use this identity A LOT when you're doing trig proofs — coming up on pages 74-75.

This is a handy thing to know — and one the examiners love testing. Basically, if you've got a trig equation with a tan in it, together with a sin or a cos — chances are you'll be better off if you rewrite the tan using this formula.

Example: Solve: $3\sin x - \tan x = 0$, for $0 \le x \le 2\pi$.

1) It's got **sin** and **tan** in it — so writing $\tan x$ as $\frac{\sin x}{\cos x}$ is probably a good move:

$$3\sin x - \tan x = 0$$

$$\Rightarrow 3\sin x - \frac{\sin x}{\cos x} = 0$$

2) Get rid of the **cos *x*** on the bottom by multiplying the whole equation by $\cos x$.

$$\Rightarrow 3\sin x\cos x - \sin x = 0$$

3) Now — there's a **common factor** of $\sin x$. Take that outside a bracket.

$$\Rightarrow \sin x\,(3\cos x - 1) = 0$$

4) And now you're almost there. You've got two things multiplying together to make zero. That means either **one or both** of them is **equal to zero**.

$$\Rightarrow \sin x = 0 \quad \text{or} \quad 3\cos x - 1 = 0$$

$\sin x = 0$

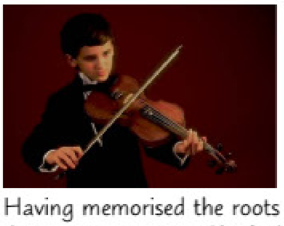
The first solution is:

$$\sin 0 = 0$$

Now find the other points where $\sin x$ is zero in the interval $0 \le x \le 2\pi$.

Remember the sin graph is zero every π radians.

$\Rightarrow x = 0, \pi, 2\pi$ radians



Having memorised the roots of sin x, smug young Sherlock had ample time to entertain his classmates as they caught up.

$3\cos x - 1 = 0$

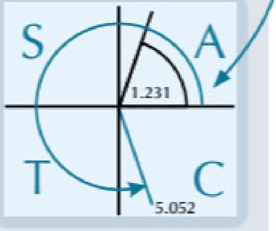
CAST gives any solutions in the interval $0 \le x \le 2\pi$.

Rearrange:

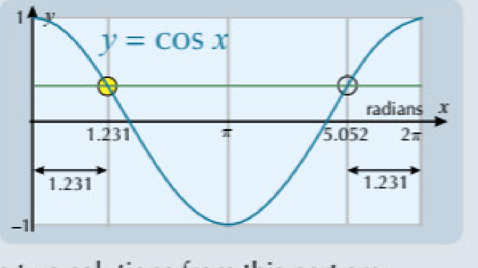
$$\cos x = \frac{1}{3}$$

So the first solution is:

$$\cos^{-1}\left(\frac{1}{3}\right) = 1.23095...$$
$$= 1.231 \text{ (3 d.p.)}$$



CAST (or the graph of $\cos x$) gives another positive solution in the 4th quadrant, where $x = 2\pi - 1.23095... = 5.052$ (3 d.p.)



And the two solutions from this part are:

$\Rightarrow x = 1.231, 5.052$ radians

So altogether you've got **five** possible solutions:

$\Rightarrow$ **$x = 0, 1.231, \pi, 5.052, 2\pi$ radians**

Be warned — you might be tempted to simplify an equation by **dividing** by a trig function. But you can **only** do this if the trig function you're dividing by is **never zero** in the range the equation is valid for. Dividing by zero is not big or clever, or even possible.

Solving Trig Equations

*And if you have **sin² x** or **cos² x**, think of this straight away...*

$$\sin^2 x + \cos^2 x \equiv 1 \Rightarrow \begin{array}{l} \sin^2 x \equiv 1 - \cos^2 x \\ \cos^2 x \equiv 1 - \sin^2 x \end{array}$$

Use this identity to get rid of a $\sin^2$ or a $\cos^2$ that's making things awkward...

Example: Solve: $2\sin^2 x + 5\cos x = 4$, for $0° \leq x \leq 360°$.

1) You can't do much while the equation's got both sin's and cos's in it.
So replace the $\sin^2 x$ bit with $1 - \cos^2 x$:

$$2(1 - \cos^2 x) + 5\cos x = 4$$ ← Now the only trig function is cos.

2) Multiply out the bracket and rearrange it so that you've got zero on one side — and you get a **quadratic** in $\cos x$:

$$\Rightarrow 2 - 2\cos^2 x + 5\cos x = 4$$
$$\Rightarrow 2\cos^2 x - 5\cos x + 2 = 0$$

3) This is a quadratic in $\cos x$. It's easier to factorise this if you make the substitution $y = \cos x$.

$$2y^2 - 5y + 2 = 0$$ ← $2y^2 - 5y + 2 = (2y \;\; ?)(y \;\; ?) = (2y - 1)(y - 2)$
$$\Rightarrow (2y - 1)(y - 2) = 0$$
$$\Rightarrow (2\cos x - 1)(\cos x - 2) = 0$$

4) Now one of the brackets must be **0**. So you get 2 equations as usual:

$2\cos x - 1 = 0$ or $\cos x - 2 = 0$

You did this example on page 62. → $\cos x = \frac{1}{2} \Rightarrow x = 60°$ and $x = 300°$ and $\cos x = 2$ ← This is a bit weird. cos x is always between −1 and 1, so you don't get any solutions from this bracket.

So at the end of all that, the only solutions you get are $x = 60°$ and $x = 300°$. How boring.

Practice Questions

Q1 a) Solve each of these equations for $0 \leq \theta \leq 2\pi$: (i) $\sin\theta = -\frac{\sqrt{3}}{2}$, (ii) $\tan\theta = -1$, (iii) $\cos\theta = -\frac{1}{\sqrt{2}}$

b) Solve each of these equations for $-180° \leq \theta \leq 180°$ (giving your answer to 1 d.p.):
(i) $\cos 4\theta = -\frac{2}{3}$ (ii) $\sin(\theta + 35°) = 0.3$ (iii) $\tan\left(\frac{1}{2}\theta\right) = 500$

Q2 Find all the solutions to $6\sin^2 x = \cos x + 5$ in the range $0 \leq x \leq 2\pi$ (answers to 3 s.f. where appropriate).

Q3 Solve $3\tan x + 2\cos x = 0$ for $-90° \leq x \leq 90°$.

Q4 Simplify: $(\sin y + \cos y)^2 + (\cos y - \sin y)^2$.

Exam Questions

Q1 a) Solve $2\cos\left(x - \frac{\pi}{4}\right) = \sqrt{3}$, for $0 \leq x \leq 2\pi$. [3 marks]

b) Solve $\sin 2x = -\frac{1}{2}$, for $0° \leq x \leq 360°$. [3 marks]

Q2 a) Show that the equation $2(1 - \cos x) = 3\sin^2 x$ can be written as $3\cos^2 x - 2\cos x - 1 = 0$. [2 marks]

b) Use this to solve the equation $2(1 - \cos x) = 3\sin^2 x$ for $0 \leq x \leq 360°$, giving your answers to 1 d.p. [6 marks]

Q3 Solve the equation $3\cos^2 x = \sin^2 x$, for $-\pi \leq x \leq \pi$. [6 marks]

Trig equations are sinful (and cosful and tanful)...

...but they are a definite source of marks — you can bet your last penny they'll be in the exam. That substitution trick to get rid of a sin² or a cos² and end up with a quadratic in sin x or cos x is a real examiners' favourite. Remember to use CAST or graphs to find all the possible solutions in the given interval, not just the one on your calculator display.