# Convex and Concave Curves

*On p.87, we briefly mentioned the third type of stationary point — the mythical "point of inflection". But before we can get to those, I should introduce you to my good friends Convex and Concave Curves (they're siblings, by the way).*

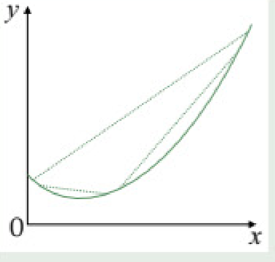
## ***Curves** can be **Convex** or **Concave***

Convex and concave are some handy words that you can use to describe the **shape** of a graph. They can be used for the **whole graph**, or just **parts** of it. It can be easy to forget which is which, so I'd pay close attention here.

### Convex

**Convex curves** are ones that curve **downwards**.

A straight line joining any two points on the curve will be **above** the curve.

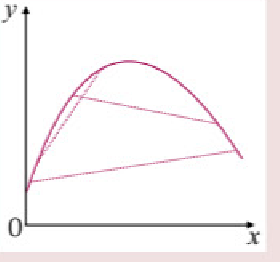



A curve is **convex** when $\mathbf{f''(x) > 0}$
i.e. when its **gradient** is **increasing**.

### Concave

**Concave curves** are ones that curve **upwards**.

A straight line joining any two points on the curve will be **below** the curve.




A curve is **concave** when $\mathbf{f''(x) < 0}$
i.e. when its **gradient** is **decreasing**.

**Example:** Find the range of $x$-values for which the graph of $y = 2x^3 - x^2 + 5$ is concave.

Start by finding the **second derivative**: $y = 2x^3 - x^2 + 5$

$$\frac{dy}{dx} = 6x^2 - 2x$$

$$\frac{d^2y}{dx^2} = 12x - 2$$

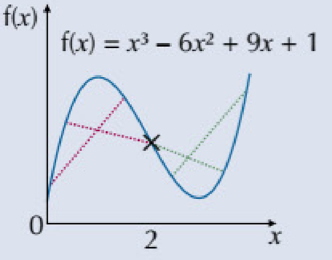
The graph is concave when the second derivative is **negative**:

$$\frac{d^2y}{dx^2} < 0 \Rightarrow 12x - 2 < 0 \Rightarrow 12x < 2 \Rightarrow x < \frac{1}{6}$$

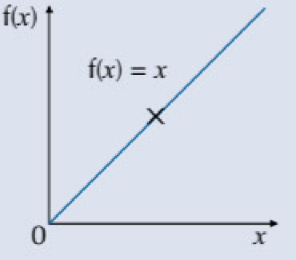
> If you're having trouble remembering which is which, here's a handy tip: concave is the one that looks like the entrance to a cave.

## ***Points of Inflection** occur when the curve **Changes** from one to the other*

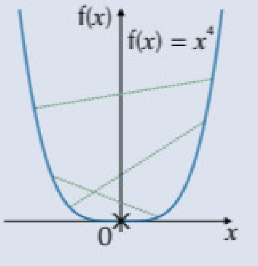
A point where the curve **changes** between concave and convex (i.e. where $f''(x)$ changes between positive and negative) is called a **point of inflection**. At a point of inflection, $\mathbf{f''(x) = 0}$ — but not all points where $f''(x) = 0$ are points of inflection. You need to look what's happening on **either side** of the point to see if the **sign** of $f''(x)$ is changing.




$f''(x) = 6x - 12$, so $f''(x) = 0$ when $x = 2$
$\mathbf{f''(x) < 0}$ for $x < 2$ and $\mathbf{f''(x) > 0}$ for $x > 2$
The curve changes from **concave** to **convex** at this point, so there is a **point of inflection** at $x = 2$.




$f''(x) = 0$ for all $x$
The curve is **never** concave or convex, so there are **no** points of inflection.




$f''(x) = 12x^2$, so
$f''(x) = 0$ when $x = 0$
but $f''(x) \geq 0$ for all $x$
The whole curve is **convex**, so (0, 0) is **not** a point of inflection.

# Convex and Concave Curves

## *Points of Inflection can also be Stationary Points*

If you have a point of inflection that also happens to be a stationary point (i.e. $f'(x) = 0$ as well), then you've got yourself what's called a **stationary point of inflection**. And the award for 'Least Imaginative Name' goes to...

**Example:** Show that the graph of $y = x^3 - 9x^2 + 27x - 26$ has a stationary point of inflection at $x = 3$.

To show that $x = 3$ is a stationary point of inflection, you need to show three things:

1) the first derivative is zero at $x = 3$,  2) the second derivative is zero at $x = 3$,
3) the second derivative changes sign either side of $x = 3$. ← i.e. the curve changes between convex and concave at this point.

So start by differentiating the function twice: $\frac{dy}{dx} = 3x^2 - 18x + 27$ and $\frac{d^2y}{dx^2} = 6x - 18$

1) When $x = 3$, $\frac{dy}{dx} = 3(3)^2 - 18(3) + 27 = 27 - 54 + 27 = 0$ — so there is a stationary point at $x = 3$.

2) When $x = 3$, $\frac{d^2y}{dx^2} = 6(3) - 18 = 18 - 18 = 0$.

3) When $x > 3$, $\frac{d^2y}{dx^2} > 0$, and when $x < 3$, $\frac{d^2y}{dx^2} < 0$ — so there is a point of inflection at $x = 3$.

So the graph has a stationary point of inflection at $x = 3$, as required.

**Example:** Find all of the stationary points of $f(x) = 3x^5 - 5x^3$, and determine their nature.

Start by finding the stationary points:

$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$, so $f'(x) = 0$ when $x^2 = 0$ or when $(x^2 - 1) = 0$, i.e. when $x = 0$ or $x = \pm 1$

$f(0) = 0$, $f(1) = -2$ and $f(-1) = 2$, so the stationary points are at $(0, 0)$, $(1, -2)$ and $(-1, 2)$.

Find the value of $f''(x)$ to determine their nature:

$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$
When $x = -1$, $f''(x) = -30 < 0$, so $(-1, 2)$ is a maximum
When $x = 1$, $f''(x) = 30 > 0$, so $(1, -2)$ is a minimum
When $x = 0$, $f''(x) = 0$, so check $f''(x)$ on either side of 0.

When $x$ is small and positive, $30x$ is positive and $(2x^2 - 1)$ is negative, so $f''(x)$ is negative.
When $x$ is small and negative, $30x$ is negative and $(2x^2 - 1)$ is negative, so $f''(x)$ is positive.

So $f''(x)$ changes sign either side of 0, i.e. $f(x)$ changes from convex to concave, meaning that $(0, 0)$ is a stationary point of inflection.

## *Practice Questions*

Q1 For each of the following intervals, determine whether $y = x^5 - 2x^4 - 3x + 2$ is concave over the whole interval, convex over the whole interval, or has a point of inflection in the interval:
a) $-1 < x < 0$,  b) $0 < x < 1$,  c) $1 < x < 2$,  d) $2 < x < 3$.

Q2 The graph of $y = 2x^3 - x^4$ has two points of inflection, one of which is a stationary point of inflection. Find the coordinates of both and determine which one is the stationary point of inflection.

**Exam Questions**

Q1 Show that $f(x) = 2x^6 - 6x^5 + 5x^4$ has no points of inflection. [5 marks]

Q2 For $f(x) = x^4 + \frac{5}{3}x^3 + x^2 - 2x$, find the ranges of $x$ for which $f(x)$ is concave and convex. [5 marks]

Q3 Find any stationary points of the curve $y = 4 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{4}x^4$, and determine their nature. [9 marks]

## *Cases of convex/concave confusion cause crying in classrooms...*

*I hope you enjoyed that, because that's the last mention of graphs for a while now. Be extra careful with those 'stationary point of inflection' questions — it's not enough to show that f'(x) and f''(x) are zero, you also need to show that f''(x) changes sign at the point. I always forgot that bit, and it got me into a whole world of trouble — turns out polishing derivatives is really dull. What I'm trying to say is: don't repeat my mistakes — check for a change of sign.*