Differentiating e^x, ln x and a^x

Remember those special little functions from Section 6? Well you're about to find out just how special they are as we take a look at how to differentiate them. I can tell you're overcome by excitement so I won't keep you waiting...

The Gradient of $y = e^x$ *is* e^x

In the last section (see p.80) you saw that 'e' was just a number for which the **gradient of e^x** was e^x — which makes it pretty simple to **differentiate**. Don't worry if you have a constant in there as well — just multiply the whole thing by the constant when you differentiate.

$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$

Example: If $f(x) = e^{x^2} + 2e^x$, find $f'(x)$ for $x = 0$.

Break down the function into its two bits and differentiate them **separately**:

$y = e^{x^2}$ and $y = 2e^x$

This is the tricky bit.
Use the **chain rule** from p.94: $u = x^2$ and $y = e^u$

Both u and y are now easy to differentiate:

$$\frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2x = e^{x^2} \times 2x = \mathbf{2xe^{x^2}}$$

This bit's easy.
If $y = 2e^x$ then $\frac{dy}{dx} = \mathbf{2e^x}$ too.

Put the bits back together and you end up with $\mathbf{f'(x) = 2xe^{x^2} + 2e^x}$.

So when $x = 0$, $f'(x) = 0 + 2e^0 = \mathbf{2}$

There's a general rule you can use when you have a **function** in the exponent (e.g. e^{x^2}):

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

You can prove this using the **chain rule**.

Turn $y = \ln x$ *into* $x = e^y$ *to Differentiate*

$$y = \ln x$$
$$\frac{dy}{dx} = \frac{1}{x}$$

You can just **learn** this result, but it comes from another bit of mathematical fiddling:

If $y = \ln x$, then $x = e^y$ (see p.80).

Differentiating gives $\frac{dx}{dy} = e^y$, and $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{e^y} = \frac{1}{x}$ (since $x = e^y$). Nice eh?

Example: Find $\frac{dy}{dx}$ if $y = \ln(x^2 + 3)$.

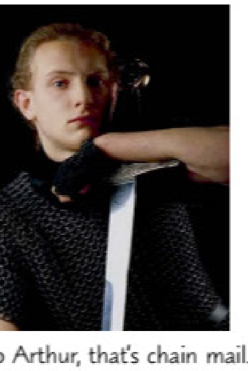
Use the **chain rule** again for this one: $y = \ln u$ and $u = x^2 + 3$.

$\frac{dy}{du} = \frac{1}{u}$ (from above) and $\frac{du}{dx} = 2x$, so: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 2x = \mathbf{\frac{2x}{x^2 + 3}}$

Just like before, there's a handy rule that you can learn for when there's a function **inside** the ln (e.g. $\ln(x^2 + 3)$):

$$y = \ln(f(x))$$
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Again, you can use the **chain rule** to prove this result.



No Arthur, that's chain mail. You're not going to differentiate anything with that.

Differentiating e^x, $\ln x$ and a^x

Learn the rule for Differentiating a^x

Here's another little rule you need to **learn**:

For any constant a,

$$\frac{d}{dx}(a^x) = a^x \ln a$$

The rule $\frac{d}{dx}(e^x) = e^x$ is actually just a special case of this rule, since $\ln e = 1$.

You can prove this rule using implicit differentiation (see p.105),
but for now you can just use the result it without worrying about where it comes from.

Example: Find the equation of the tangent to the curve $y = 3^{-2x}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Use the **chain rule** to find $\frac{dy}{dx}$: $u = -2x$ and $y = 3^u \Rightarrow \frac{du}{dx} = -2$ and $\frac{dy}{du} = 3^u \ln 3$ (using the rule above)

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3^u \ln 3 \times -2 = -2(3^{-2x} \ln 3)$$

Now you can find the equation of the **tangent**: At $\left(\frac{1}{2}, \frac{1}{3}\right)$, $\frac{dy}{dx} = -2(3^{-2x} \ln 3) = -\frac{2}{3} \ln 3$

Using the equation $y - y_1 = m(x - x_1)$:

$$y - \frac{1}{3} = \left(-\frac{2}{3}\ln 3\right)\left(x - \frac{1}{2}\right) \Rightarrow 3y - 1 = \ln 3 - (2 \ln 3)x$$

$$\Rightarrow (2 \ln 3)x + 3y - (1 + \ln 3) = 0$$

This is in the form $ax + by + c = 0$, but any of the forms from p.36 would be fine.

Practice Questions

Q1 Find $\frac{dy}{dx}$ when a) $y = e^{5x^2}$ b) $y = \ln(6 - x^2)$ c) $x = 2e^y$ d) $x = \ln(2y + 3)$ e) $y = 10^x$ f) $x = 5^y$

Q2 a) Find the derivative, with respect to x, of $f(x) = 3^x + 4x$.

b) Hence find the derivative, with respect to x, of $g(x) = \ln(3^x + 4x)$.

Q3 a) Differentiate $y = 2^{-2x}$ with respect to x.

b) Find the gradient of this curve when $x = 1$.

c) Hence find the equation of the normal to the curve at $x = 1$.

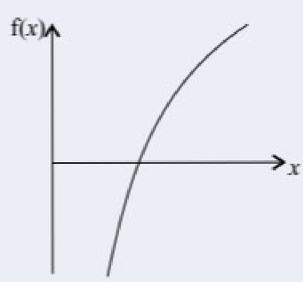
Exam Questions

Q1 Differentiate the following with respect to x.

a) $\sqrt{e^x + e^{2x}}$ [3 marks]

b) $3e^{2x+1} - \ln(1 - x^2) + 2x^3$ [3 marks]

Q2 A sketch of the function $f(x) = 4 \ln 3x$ is shown in the diagram on the right.



a) Find $f'(x)$ at the point where $x = 1$. [3 marks]

b) Find the equation of the tangent to the curve at the point $x = 1$. [3 marks]

Q3 a) Curve A has the equation $y = 4^x$.
What are the coordinates of the point on A where $\frac{dy}{dx} = \ln 4$? [2 marks]

b) Curve B has the equation $y = 4^{(x-4)^3}$.
Find the gradient of B at the point $\left(3, \frac{1}{4}\right)$. [4 marks]

This is a topic for lumberjacks — it's all about logs and a^xes...

Well, I don't know about you but my heart is still racing from all that excitement. Doesn't it feel nice to differentiate something that isn't some boring power of x? Don't worry, you can thank me later — there's plenty more to learn first.

Differentiating sin, cos and tan

So you think you know all there is to know about trigonometry. Well think again, 'cos here it comes again. (You see what I did there with the 'cos'? Pun #27 from 'Ye Olde Booke of Maths Punnes'...)

The **Rules** for differentiating **sin**, **cos** and **tan** only work in **Radians**

For **trigonometric functions**, where the angle is measured in **radians** (see p.56), the following rules apply:

If $y =$		$\frac{dy}{dx} =$
$\sin x$	$\longrightarrow$	$\cos x$
$\cos x$	$\longrightarrow$	$-\sin x$
$\tan x$	$\longrightarrow$	$\sec^2 x$

The derivative of tan is given on the formula sheet.

There's loads more about sec (and cosec and cot) on p.66-67.

You can use the chain rule to show that, if k is a constant:

$\sin kx \longrightarrow k \cos kx$

$\cos kx \longrightarrow -k \sin kx$

$\tan kx \longrightarrow k \sec^2 kx$

It's handy to know these so you don't have to write out the chain rule every time.

Use the **Chain Rule** with **sin/cos/tan (f(x))**

Example: Differentiate $y = \cos 2x + \sin(x + 1)$ with respect to x.

It's the **chain rule** (again) for both parts of this equation:

1) Differentiate $y = \cos 2x$: $y = \cos u$, $u = 2x$,

so $\frac{dy}{du} = -\sin u$ (see above) and $\frac{du}{dx} = 2 \Rightarrow \frac{dy}{dx} = -2\sin 2x$

This is the result in the box above, with k = 2.

2) Differentiate $y = \sin(x + 1)$: $y = \sin u$, $u = x + 1$,

so $\frac{dy}{du} = \cos u$ (see above) and $\frac{du}{dx} = 1 \Rightarrow \frac{dy}{dx} = \cos(x + 1)$

3) Put it all together to get: $\frac{dy}{dx} = \mathbf{-2\sin 2x + \cos(x + 1)}$

Example: Find $\frac{dy}{dx}$ for $x = \tan 3y$ at the point $\left(1, \frac{\pi}{12}\right)$.

You can just write this down if you know the result — you don't have to write out the chain rule.

1) First use $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ to find $\frac{dy}{dx}$: $x = \tan 3y \Rightarrow \frac{dx}{dy} = 3\sec^2 3y \Rightarrow \frac{dy}{dx} = \frac{1}{3\sec^2 3y} = \mathbf{\frac{1}{3}\cos^2 3y}$

2) Now **substitute** $y = \frac{\pi}{12}$: $\frac{dy}{dx} = \frac{1}{3}\cos^2\frac{\pi}{4} = \frac{1}{3} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{3 \times 2} = \mathbf{\frac{1}{6}}$

The rule $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ was on p.94 if you want a reminder.

Remember to use **Trig Identities** where **Necessary**

Example: For $y = 2\cos^2 x + \sin 2x$, show that $\frac{dy}{dx} = 2(\cos 2x - \sin 2x)$.

1) Writing out the equation in a **slightly different way** helps with the chain rule: $y = 2(\cos x)^2 + \sin 2x$.

2) For the first bit, $y = 2u^2$, $u = \cos x$, so $\frac{dy}{du} = 4u$ and $\frac{du}{dx} = -\sin x$.

For the second bit, $y = \sin u$, $u = 2x$, so $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2$.

You could also use the identity $\cos 2x \equiv 2\cos^2 x - 1$ before differentiating. You'll get the same answer.

3) Putting it all in the chain rule formula gives $\frac{dy}{dx} = -4\sin x \cos x + 2\cos 2x$.

4) From the target answer in the question it looks like we need a $\sin 2x$ from somewhere, so use the **double angle formula** (see p.71) $\sin 2x \equiv 2\sin x \cos x$:

$\frac{dy}{dx} = -2\sin 2x + 2\cos 2x$, which **rearranges** nicely to give $\frac{dy}{dx} = \mathbf{2(\cos 2x - \sin 2x)}$. Et voilà.

Differentiating sin, cos and tan

Use the **Chain Rule** for **Combinations** of functions

You should be able to use the chain rule to differentiate functions that are combinations of any of the functions from this section. So make sure you haven't forgotten the rules for differentiating exponentials and logs already.

Example: Find $f'(x)$, where $f(x) = e^{\cos 3x}$.

Use the **chain rule**: $y = e^u$ and $u = \cos 3x$, so $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = -3\sin 3x$

So $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times (-3\sin 3x) = -3e^u \sin 3x = \mathbf{-3e^{\cos 3x}\sin 3x}$

You can **Differentiate sin** and **cos** from **First Principles**

You saw how to differentiate a function from **first principles** back on p.86. You can do this for sin and cos too, but you'll need to dust off your **small angle approximations** (p.69) and your **addition formulas** (p.70).

Example: Differentiate $f(x) = \sin x$ from first principles.

Start by writing out the formula: $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

Substitute $f(x) = \sin x$: $= \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin x}{h}\right)$

Expand $\sin(x + h)$ with the **addition formula**: $= \lim_{h\to 0}\left(\frac{(\sin x\cos h + \cos x \sin h)-\sin x}{h}\right)$

It's helpful to collect the $\sin x$ and $\cos x$ terms here: $= \lim_{h\to 0}\left(\frac{\sin x(\cos h - 1) + \cos x(\sin h)}{h}\right)$

You're interested in when h gets really small (as $h \to 0$), so you can use the **small angle approximations** ($\sin h \approx h$, $\cos h \approx 1 - \frac{1}{2}h^2$): $= \lim_{h\to 0}\left(\frac{\left(-\frac{1}{2}h^2\right)\sin x + h\cos x}{h}\right)$

h appears on the top and bottom of the fraction, so cancel it: $= \lim_{h\to 0}\left(-\frac{1}{2}h\sin x + \cos x\right)$

As $h \to 0$, $\frac{1}{2}h\sin x \to 0$, so it disappears: $= \mathbf{\cos x}$

Practice Questions

When studying nest architecture, a solid understanding of twigonometry and differeggtiation is essential.

Q1 Find $f'(x)$ for the following functions:

a) $f(x) = 2\cos 3x$ b) $f(x) = \sqrt{\tan x}$ c) $f(x) = \cos(e^x) + e^{\sin x}$

Q2 Differentiate $\sin^2(x + 2)$ with respect to x.

Exam Questions

Q1 Find the gradient of the tangent to the curve $y = \sin^2 x - 2\cos 2x$ at the point where $x = \frac{\pi}{12}$ radians. [4 marks]

Q2 Find the equation of the normal to the curve $x = \sin 4y$ that passes through the point $\left(0, \frac{\pi}{4}\right)$. Give your answer in the form $y = mx + c$, where m and c are constants to be found. [6 marks]

Q3 By differentiating from first principles, prove that the derivative of $\cos x$ is $-\sin x$. [5 marks]

I'm having an identity crisis — I can't differentiate between sin and cos...

Don't get tied down by the chain rule (pun #28...). After a bit of practice you'll be able to do it a lot quicker in one step — just say in your working 'using the chain rule...' so the examiner can see how clever you are.

Chain Rule

Now it's time to upgrade your differentiation with some new exciting features. And about time too — I bet you were starting to get bored of doing the same old 'multiply by the power, reduce it by one' rigmarole every time.

*The **Chain Rule** is used for **Functions of Functions***

The **chain rule** is a nifty little tool that allows you to differentiate complicated functions by **splitting them up** into easier ones. The trick is spotting **how** to split them up, and choosing the right bit to **substitute**.

Chain Rule Method

- Pick a suitable function of x for 'u' and rewrite y in terms of u.
- Differentiate u (with respect to x) to get $\frac{du}{dx}$, and differentiate y (with respect to u) to get $\frac{dy}{du}$.
- Stick it all in the formula.

If $y = f(u)$ and $u = g(x)$ then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example: Find the exact value of $\frac{dy}{dx}$ when $x = 1$ for $y = \frac{1}{\sqrt{x^2 + 4x}}$.

First, write y in terms of **powers** to make it easier to differentiate: $y = (x^2 + 4x)^{-\frac{1}{2}}$.

Pick a chunk of the equation to call 'u', and rewrite y **in terms of u** — in this case let $u = x^2 + 4x$, so $y = u^{-\frac{1}{2}}$.

Now differentiate both bits **separately**: $u = x^2 + 4x \Rightarrow \frac{du}{dx} = 2x + 4$ and $y = u^{-\frac{1}{2}} \Rightarrow \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$

Use the **chain rule** to find $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \times (2x + 4)$

Substitute in $u = x^2 + 4x$ and **rearrange**: $\frac{dy}{dx} = -\frac{1}{2}(x^2 + 4x)^{-\frac{3}{2}} \times (2x + 4) = -\frac{x + 2}{(\sqrt{x^2 + 4x})^3}$

Finally, put in $x = 1$ to get the answer: $\frac{dy}{dx} = -\frac{1 + 2}{(\sqrt{1^2 + 4})^3} = -\frac{3}{5\sqrt{5}} = -\frac{3\sqrt{5}}{25}$ ← 'Exact' means leave in surd form where necessary.

*Use **dy/dx = 1 ÷ dx/dy** for **x = f(y)***

The **principle** of the chain rule can also be used where **x is given in terms of y** (i.e. $x = f(y)$).
$\frac{dy}{dx} \times \frac{dx}{dy} = \frac{dy}{dy} = 1$, so rearranging gives this formula.

For $x = f(y)$, use

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

Example: A curve has the equation $x = y^3 + 2y - 7$. Find $\frac{dy}{dx}$ at the point $(-4, 1)$.

Forget that the x's and y's are in the 'wrong' places and differentiate as usual: $\frac{dx}{dy} = 3y^2 + 2$

Use $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ to find $\frac{dy}{dx}$ at $(-4, 1)$: $\frac{dy}{dx} = \frac{1}{3y^2 + 2} \Rightarrow$ when $y = 1$, $\frac{dy}{dx} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$

$\frac{dy}{dx}$ isn't a fraction, but it acts a bit like one here.

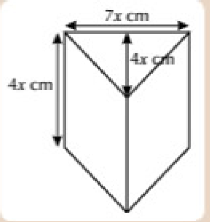
*The **Chain Rule** lets you **Connect** different **Rates of Change***

1) Some situations have a number of **linked variables**, like length, surface area and volume, or distance, speed and acceleration.
2) If you know the rate of change of **one** of these linked variables, and the **equations that connect** the variables, you can use the chain rule to help you find the rate of change of the **other variables**.
3) There might be a **hidden derivative** given in words, so keep an eye out for words like '**rate**' or '**per**'. Watch out — if you're told that something is '**decreasing**' or something similar, the rate might be **negative**.
4) Make sure you know whether the question is asking for $\frac{dy}{dx}$ or $\frac{dx}{dy}$ — use the rule above if you need to.

Chain Rule

This is one of those topics where the most awkward bit is **getting your head round** the information in the question. The actual maths is **nowhere near** as bad as the questions usually make it sound. Honest.

Example: A scientist is testing how a new material expands when it is gradually heated. The diagram shows the sample being tested, which is shaped like a triangular prism. After t minutes, the triangle that forms the base of the prism has base length $7x$ cm and height $4x$ cm, and the height of the prism is also $4x$ cm.
If the sample expands at a constant rate, given by $\frac{dx}{dt} = 0.05$ cm min^{-1}, find an expression in terms of x for $\frac{dV}{dt}$, where V is the volume of the prism.



The best way to start this kind of question is to write down what you know. There's enough information to write an expression for the volume of the prism: $V = \left(\frac{1}{2} \times 7x \times 4x\right) \times 4x = 56x^3$ cm^3

Differentiate this with respect to x: $\frac{dV}{dx} = 168x^2$

You know that $\frac{dx}{dt} = 0.05$. So use the **chain rule** to find $\frac{dV}{dt}$: $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 168x^2 \times 0.05 = \mathbf{8.4x^2}$

Example: A giant metal cube from space is cooling after entering the Earth's atmosphere. As it cools, the surface area of the cube decreases at a constant rate of 0.027 m^2 s^{-1}.
If the side length of the cube after t seconds is x m, find $\frac{dx}{dt}$ at the point when $x = 15$ m.

The cube has side length x m, so the surface area of the cube is $A = 6x^2 \Rightarrow \frac{dA}{dx} = 12x$

A decreases at a constant rate of 0.027 m^2 s^{-1} — we can write this as $\frac{dA}{dt} = -0.027$ ← This value is negative because A is decreasing.

Now use the **chain rule** to find $\frac{dx}{dt}$: $\frac{dx}{dt} = \frac{dx}{dA} \times \frac{dA}{dt} = \frac{1}{\left(\frac{dA}{dx}\right)} \times \frac{dA}{dt} = \frac{1}{12x} \times -0.027 = -\frac{0.00225}{x}$

So when $x = 15$, $\frac{dx}{dt} = -\frac{0.00225}{x} = -\frac{0.00225}{15} = \mathbf{-0.00015\ m\,s^{-1}}$

Practice Questions

Q1 Differentiate with respect to x: a) $y = \sqrt{x^3 + 2x^2}$ b) $y = \frac{1}{\sqrt{x^3 + 2x^2}}$

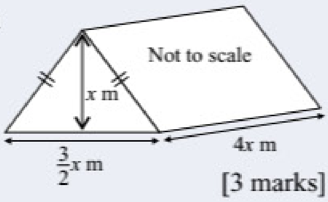
Q2 A cuboid of length x cm, width $2x$ cm and height $3x$ cm is expanding, for some unexplained reason. If A is its surface area and V is its volume, find $\frac{dA}{dx}$ and $\frac{dV}{dx}$, and hence show that if $\frac{dV}{dt} = 3$, then $\frac{dA}{dt} = \frac{22}{3x}$.

Exam Questions

Q1 a) Find $\frac{dy}{dx}$ for the curve given by the equation $x = \sqrt{y^2 + 3y}$ at the point (2, 1). [5 marks]

b) Hence find the equation of the tangent to the curve at (2, 1), in the form $y = mx + c$. [2 marks]

Q2 The triangular prism shown in the diagram is expanding. The dimensions of the prism after t seconds are given in terms of x. The prism is $4x$ m long, and its cross-section is an isosceles triangle with base $\frac{3}{2}x$ m and height x m.



a) Show that, if the surface area of the prism after t seconds is A m^2, then $A = \frac{35}{2}x^2$. [3 marks]

The surface area of the prism is increasing at a constant rate of 0.07 m^2 s^{-1}.

b) Find $\frac{dx}{dt}$ when $x = 0.5$. [3 marks]

c) If the volume of the prism is V m^3, find the rate of change of V when $x = 1.2$. [4 marks]

I'm in the middle of a chain rule differentiation...

If you get stuck on a question like this, don't panic. Somewhere in the question there'll be enough information to write at least one equation linking some of the variables. If in doubt, write down any equations you can make, differentiate them, and see which of the resulting expressions you can link using the chain rule to make the thing you're looking for.

Product and Quotient Rules

In maths-speak, multiplying two things gives you a 'product' and dividing them gives you a 'quotient'. And since the world of maths is a beautiful, harmonious place full of natural symmetry, there's a rule for differentiating each.

*Use the **Product Rule** to differentiate **Two Functions Multiplied Together***

This is what it looks like:

If $y = u(x)v(x)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

(where u and v are functions of x)

And here's how to use it:

Examples: Differentiate the following with respect to x:

a) $x^3 \tan x$

1) The crucial thing is to write down everything in **steps**. Start with **identifying** 'u' and 'v':
$u = x^3$ and $v = \tan x$
2) Now differentiate these two **separately**:
$\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = \sec^2 x$
3) Very carefully put all the bits into the **formula**:
$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^3 \times \sec^2 x) + (\tan x \times 3x^2)$
4) Finally, **rearrange** to make it look nicer:
$$\frac{dy}{dx} = x^3 \sec^2 x + 3x^2 \tan x$$

b) $e^{2x}\sqrt{2x-3}$

1) Again, start with **identifying** 'u' and 'v':
$u = e^{2x}$ and $v = \sqrt{2x-3}$
2) Each of these needs the **chain rule** to differentiate:
$\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = \frac{1}{\sqrt{2x-3}}$ (do it in steps if you need to...)
3) Put it all into the product rule **formula**:
$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \left(e^{2x} \times \frac{1}{\sqrt{2x-3}}\right) + (\sqrt{2x-3} \times 2e^{2x})$
4) **Rearrange** and **simplify**:
$$\frac{dy}{dx} = e^{2x}\left(\frac{1}{\sqrt{2x-3}} + 2\sqrt{2x-3}\right) = e^{2x}\left(\frac{1+2(2x-3)}{\sqrt{2x-3}}\right)$$
$$= \frac{e^{2x}(4x-5)}{\sqrt{2x-3}}$$

*Use the rules **Together** to differentiate **Complicated Functions***

Example: Solve the equation $\frac{d}{dx}((x^3 + 3x^2)\ln x) = 2x^2 + 5x$, leaving your answer as an exact value of x.

1) Since $(x^3 + 3x^2)\ln x$ is a product of two functions, use the **product rule**:
$u = x^3 + 3x^2 \Rightarrow \frac{du}{dx} = 3x^2 + 6x$ and $v = \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$ (see p.96)
So $\frac{d}{dx}((x^3 + 3x^2)\ln x) = \left[(x^3 + 3x^2) \times \frac{1}{x}\right] + [\ln x \times (3x^2 + 6x)] = x^2 + 3x + (3x^2 + 6x)\ln x$.
2) Now put this into the **equation** from the question in place of $\frac{d}{dx}((x^3 + 3x^2)\ln x)$:
$x^2 + 3x + (3x^2 + 6x)\ln x = 2x^2 + 5x$
3) **Rearrange** and **solve** as follows:
$(3x^2 + 6x)\ln x = 2x^2 + 5x - x^2 - 3x \Rightarrow (3x^2 + 6x)\ln x = x^2 + 2x$
$\Rightarrow \ln x = \frac{x^2 + 2x}{3(x^2 + 2x)} = \frac{1}{3} \Rightarrow x = e^{\frac{1}{3}}$

You're asked for an exact value so leave in terms of e.

*Use the **Quotient Rule** for one function **Divided By** another*

A **quotient** is one function **divided by** another one.
The **rule** for differentiating quotients looks like this:

If $y = \frac{u(x)}{v(x)}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

You could, if you wanted to, just use the **product rule** on $y = uv^{-1}$ (try it — you'll get the same answer).
This way is so much **quicker** and **easier** though — and it's on the **formula sheet**.

Product and Quotient Rules

The quotient rule might look a little ugly, but it's lovely when you get to know it. That's why I call it Quotimodo...

Example: Find the gradient of the tangent to the curve with equation $y = \frac{(2x^2-1)}{(3x^2+1)}$, at the point (1, 0.25).

1) '**Gradient**' means **differentiate**, so identify u and v for the **quotient rule**, and differentiate **separately**:

$$u = 2x^2 - 1 \Rightarrow \frac{du}{dx} = 4x \quad \text{and} \quad v = 3x^2 + 1 \Rightarrow \frac{dv}{dx} = 6x$$

2) It's very important that you get things in the **right order**, so concentrate on what's going where:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(3x^2+1)(4x)-(2x^2-1)(6x)}{(3x^2+1)^2}$$

3) Now you can **simplify** things:

$$\frac{dy}{dx} = \frac{x[4(3x^2+1)-6(2x^2-1)]}{(3x^2+1)^2} = \frac{x[12x^2+4-12x^2+6]}{(3x^2+1)^2} = \frac{10x}{(3x^2+1)^2}$$

4) Finally, put in $x = 1$ to find the **gradient** at (1, 0.25): $\frac{dy}{dx} = \frac{10}{(3+1)^2} = \mathbf{0.625}$

*Find **Further Rules** using the **Quotient Rule***

Example: Use the quotient rule to differentiate $y = \frac{\cos x}{\sin x}$, and hence show that for $y = \cot x$, $\frac{dy}{dx} = -\text{cosec}^2\, x$.

1) Start off by **identifying** u and v: $u = \cos x$ and $v = \sin x \Rightarrow \frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = \cos x$ (see p.98)

2) Putting everything in the quotient rule **formula** gives:

$$\frac{dy}{dx} = \frac{(\sin x \times -\sin x)-(\cos x \times \cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

3) Use a **trig identity** to simplify this ($\sin^2 x + \cos^2 x \equiv 1$ should do the trick...):

$$\frac{dy}{dx} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

There's more on cosec, sec and cot on the next page.

4) Linking this back to the question, since $\tan x = \frac{\sin x}{\cos x}$, and $\cot x = \frac{1}{\tan x}$, then $y = \frac{\cos x}{\sin x} = \cot x$.

And since $\text{cosec}\, x = \frac{1}{\sin x}$, $\frac{dy}{dx} = -\frac{1}{\sin^2 x} = \mathbf{-cosec^2\, x}$. QED*

*Quite Exciting Differentiation

Practice Questions

Q1 Find the value of the gradient for: a) $y = e^{2x}(x^2 - 3)$ when $x = 0$, b) $y = \ln x \sin x$ when $x = 1$.

Q2 Find the equation of the tangent to the curve $y = \frac{6x^2+3}{4x^2-1}$ at the point (1, 3).

Exam Questions

Q1 Find $\frac{dy}{dx}$ for each of the following functions. Simplify your answer where possible.

a) $y = \ln(3x+1)\sin(3x+1)$ [4 marks]

b) $y = \frac{\sqrt{x^2+3}}{\cos 3x}$ [4 marks]

Q2 Given that $y = \frac{e^x + x}{e^x - x}$, find $\frac{dy}{dx}$ when $x = 0$. [3 marks]

It's not my fault that I love maths — I'm a product of my environment...

Sing along with me: "A function of a function wants the — chain rule. A function times a function wants the — product rule. A function over a function wants the — quotient rule. A rate of change is connected to the — knee bone..."

More Differentiation

Now that you're a master of the product and quotient rules, there are some more functions you can differentiate. Can you feel your power growing? Soon, no function will be able to stand against you and your mighty calculus.

d/dx of **cosec**, **sec** and **cot** come from the **Quotient Rule**

Since **cosec**, **sec** and **cot** are just the **reciprocals** of sin, cos and tan, the quotient rule can be used to differentiate them. The results are on the formula sheet, but it will help a lot if you can show **where they come from**.

$$y = \operatorname{cosec} x = \frac{1}{\sin x}$$

1) For the quotient rule:
$u = 1 \Rightarrow \frac{du}{dx} = 0$ and $v = \sin x \Rightarrow \frac{dv}{dx} = \cos x$

2) $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$= \frac{(\sin x \times 0) - (1 \times \cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}$

3) Since $\cot x = \frac{\cos x}{\sin x}$, and $\operatorname{cosec} x = \frac{1}{\sin x}$,
$\frac{dy}{dx} = -\frac{\cos x}{\sin x} \times \frac{1}{\sin x} = \mathbf{-\operatorname{cosec} x \cot x}$

$$y = \sec x = \frac{1}{\cos x}$$

1) For the quotient rule:
$u = 1 \Rightarrow \frac{du}{dx} = 0$ and $v = \cos x \Rightarrow \frac{dv}{dx} = -\sin x$

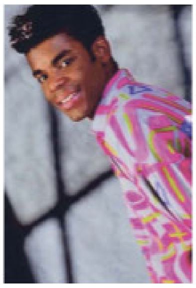
2) $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$= \frac{(\cos x \times 0) - (1 \times -\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$

3) Since $\tan x = \frac{\sin x}{\cos x}$, and $\sec x = \frac{1}{\cos x}$,
$\frac{dy}{dx} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \mathbf{\sec x \tan x}$

If $y =$		$\frac{dy}{dx} =$
$\operatorname{cosec} x$	$\longrightarrow$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\longrightarrow$	$\sec x \tan x$
$\cot x$	$\longrightarrow$	$-\operatorname{cosec}^2 x$

Go back a page for this one. Have a go at writing it out like the ones above, starting with $y = \frac{\cos x}{\sin x}$.

If you can't remember which trig functions give a negative result when you differentiate them, just remember it's all the ones that begin with c — cos, cosec and cot.



I'm too sec-sy for my shirt, cosec-sy it hurts...

Use the **Chain**, **Product** and **Quotient Rules** with **cosec**, **sec** and **cot**

Once you're familiar with the three rules in the box above you can use them with the **chain**, **product** and **quotient** rules and in combination with all the **other functions** we've seen so far.

Examples: Find $\frac{dy}{dx}$ for the following functions:

a) $y = \sec(2x^2)$

This is a function of a function, so think 'chain rule':
$y = \sec u$ and $u = 2x^2$

Differentiate y and u:
$\frac{dy}{du} = \sec u \tan u$ (see above)
$= \sec(2x^2)\tan(2x^2)$
$\frac{du}{dx} = 4x$

Then put these into the formula:
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \mathbf{4x \sec(2x^2)\tan(2x^2)}$

b) $y = e^x \cot x$

This is a product of two functions, so think 'product rule':
$u = e^x$ and $v = \cot x$

Differentiate u and v:
$\frac{du}{dx} = e^x$
$\frac{dv}{dx} = -\operatorname{cosec}^2 x$ (see above)

Then put these into the formula:
$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (e^x \times -\operatorname{cosec}^2 x) + (\cot x \times e^x)$
$= \mathbf{e^x(\cot x - \operatorname{cosec}^2 x)}$

More Differentiation

The **Rules** might need to be used **Twice**

Some questions will really stretch your alphabet with a multitude of *u*'s and *v*'s:

Example: Differentiate $y = e^x \tan^2 3x$

1) First off, this is a **product**, so use the **product rule**: $u = e^x$ (so $\frac{du}{dx} = e^x$) and $v = \tan^2 3x$
2) To find $\frac{dv}{dx}$ for the product rule, you need the **chain rule**:

   $v = u_1^2$, where $u_1 = \tan 3x$

   $\frac{dv}{du_1} = 2u_1 = 2\tan 3x$ and $\frac{du_1}{dx} = 3\sec^2 3x \Rightarrow \frac{dv}{dx} = 6\tan 3x \sec^2 3x$
3) Now you can put this result in the product rule formula to get $\frac{dy}{dx}$:

$$\frac{dy}{dx} = (\underbrace{e^x}_{u} \times \underbrace{6\tan 3x \sec^2 3x}_{\frac{dv}{dx}}) + (\underbrace{\tan^2 3x}_{v} \times \underbrace{e^x}_{\frac{du}{dx}}) = \mathbf{e^x \tan 3x\,(6\sec^2 3x + \tan 3x)}$$

Differentiate **Again** for d^2y/dx^2, **Turning Points**, **Stationary Points** etc.

Example: Determine the nature of the stationary point of the curve $y = \frac{\ln x}{x^2}$ $(x > 0)$.

1) First use the **quotient rule** to find $\frac{dy}{dx}$: $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$, $v = x^2 \Rightarrow \frac{dv}{dx} = 2x$. So $\frac{dy}{dx} = \frac{1 - 2\ln x}{x^3}$.
2) The stationary points occur where $\frac{dy}{dx} = 0$: $\frac{1 - 2\ln x}{x^3} = 0 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{\frac{1}{2}}$
3) To find out if it's a maximum or minimum, differentiate $\frac{dy}{dx}$ (using the quotient rule again) to get $\frac{d^2y}{dx^2}$:

   $u = 1 - 2\ln x \Rightarrow \frac{du}{dx} = -\frac{2}{x}$, $v = x^3 \Rightarrow \frac{dv}{dx} = 3x^2$. So $\frac{d^2y}{dx^2} = \frac{6\ln x - 5}{x^4}$.

   Put all the bits into the quotient rule formula for yourself to make sure you're happy where these come from.
4) When $x = e^{\frac{1}{2}}$, $\frac{d^2y}{dx^2} < 0$ (i.e. **negative**), which means it's a **maximum point** (see p.87).

Practice Questions

Q1 Differentiate $f(x) = \sec(4x) - \cot(x + 1)$ with respect to x.

Q2 Find $\frac{dy}{dx}$ when $x = 0$ for $y = \operatorname{cosec}(3x - 2)$.

Exam Questions

Q1 Find $\frac{dy}{dx}$ for $y = \sin^3(2x^2)$. Simplify your answer where possible. [3 marks]

Q2 A curve with equation $y = e^x \sin x$ has 2 turning points in the interval $-\pi \le x \le \pi$.

a) Find the value of x at each of these turning points. [6 marks]

b) Determine the nature of each of the turning points. [3 marks]

Q3 a) Show that, if $f(x) = \cot x$, then $f''(x) = \frac{2\cos x}{\sin^3 x}$. [5 marks]

b) Hence show that $\left(\frac{\pi}{2}, 0\right)$ is a point of inflection of the graph of $y = \cot x$. [3 marks]

Differentiation rule #33 047 — always wear nice socks when differentiating...

Whew — they sure can pack a lot of rules into one question, can't they? Well, the good news is that the formula sheet contains lots of helpful tidbits, like the derivatives of cosec, sec and cot. It's not as helpful as this book, though.