Implicit Differentiation

This really isn't as complicated as it looks... in fact, I think you'll find that if something's implicit between x and y, it can be ximplicity itself. No, that's not a typo, it's a hilarious joke... 'implicit' between 'x' and 'y'... do you see?...

You need *Implicit Differentiation* if you *Can't* write the equation as *y = f(x)*

1) An '**implicit relation**' is the maths name for any equation in x and y that's written in the form $f(x, y) = g(x, y)$ instead of $y = f(x)$.

$f(x, y)$ and $g(x, y)$ don't actually both have to include x and y — one of them could even be a constant.

2) Some implicit relations are either awkward or impossible to rewrite in the form $\mathbf{y = f(x)}$. This can happen, for example, if the equation contains a number of **different powers of y**, or terms where **x is multiplied by y**.

3) This can make implicit relations tricky to **differentiate** — the solution is **implicit differentiation**:

Implicit Differentiation

To find $\frac{dy}{dx}$ for an implicit relation between x and y:

1) Differentiate terms in **x only** (and constant terms) with respect to x, as normal.

2) Use the **chain rule** to differentiate terms in **y only**:

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\frac{dy}{dx}$$

In other words, 'differentiate with respect to y, and stick a $\frac{dy}{dx}$ on the end'.

3) Use the **product rule** to differentiate terms in **both x and y**:

$$\frac{d}{dx}u(x)v(y) = u(x)\frac{d}{dx}v(y) + v(y)\frac{d}{dx}u(x)$$

4) **Rearrange** the resulting equation in x, y and $\frac{dy}{dx}$ to **make $\frac{dy}{dx}$ the subject.**

This version of the product rule is slightly different from the one on p.100 — here, v is a function of y, not x.

This is explicit differentiation, but... well... you'll learn about that when you're older.

This might sound complicated, but don't worry — all it means really is 'differentiate like normal, but multiply by $\frac{dy}{dx}$ every time you differentiate a y-term'. Watch out for terms in x and y where you have to use the product rule — but if you're happy with everything from p.100, you should be able to handle it.

Example: Use implicit differentiation to find $\frac{dy}{dx}$ if $2x^2y + y^3 = 6x^2 + 5$.

You need to **differentiate each term** of the equation with respect to x.

Start by sticking '$\frac{d}{dx}$' in front of each term:

$$\frac{d}{dx}2x^2y + \frac{d}{dx}y^3 = \frac{d}{dx}6x^2 + \frac{d}{dx}5$$

First, deal with the **terms in x** and **constant terms** — in this case that's the two terms on the RHS:

$$\frac{d}{dx}2x^2y + \frac{d}{dx}y^3 = \mathbf{12x + 0}$$

Now use the **chain rule** on the **term in y**:

$$\frac{d}{dx}2x^2y + 3y^2\frac{dy}{dx} = 12x$$

Using the chain rule from the box above, $f(y) = y^3$.

Leave this $\frac{dy}{dx}$ here for now.

Use the **product rule** on the term in **x and y**:

$$2x^2\frac{d}{dx}(y) + y\frac{d}{dx}(2x^2) + 3y^2\frac{dy}{dx} = 12x$$

$$2x^2\frac{dy}{dx} + y4x + 3y^2\frac{dy}{dx} = 12x$$

So in terms of the box above, $u(x) = 2x^2$ and $v(y) = y$.

You get a $\frac{dy}{dx}$ term here too (from the '$\frac{d}{dx}v(y)$' bit).

Finally, **rearrange** to make $\frac{dy}{dx}$ the subject:

$$\frac{dy}{dx}(2x^2 + 3y^2) = 12x - 4xy$$

$$\frac{dy}{dx} = \frac{12x - 4xy}{2x^2 + 3y^2}$$

Implicit Differentiation

Implicit Differentiation *still gives you an expression for the* ***Gradient***

Most **implicit differentiation** questions aren't really that different at heart to any other **differentiation question**. Once you've got an expression for the **gradient**, you'll have to **use it** to do the sort of stuff you'd normally expect.

Example: Curve A has the equation $x^2 + 2xy - y^2 = 10x + 4y - 21$

a) Find $\frac{dy}{dx}$ and show that when $\frac{dy}{dx} = 0$, $y = 5 - x$.

b) Hence find the coordinates of the stationary points of A.

a) For starters, you're going to need to find $\frac{dy}{dx}$ by **implicit differentiation**:

$$\frac{d}{dx}x^2 + \frac{d}{dx}2xy - \frac{d}{dx}y^2 = \frac{d}{dx}10x + \frac{d}{dx}4y - \frac{d}{dx}21$$

$$2x + \frac{d}{dx}2xy - \frac{d}{dx}y^2 = 10 + \frac{d}{dx}4y - 0$$

Differentiate x^2, $10x$ and 21 with respect to x.

$$2x + \frac{d}{dx}2xy - 2y\frac{dy}{dx} = 10 + 4\frac{dy}{dx}$$

Use the chain rule to differentiate y^2 and $4y$.

$$2x + 2x\frac{dy}{dx} + y\frac{d}{dx}2x - 2y\frac{dy}{dx} = 10 + 4\frac{dy}{dx}$$

$$2x + 2x\frac{dy}{dx} + 2y - 2y\frac{dy}{dx} = 10 + 4\frac{dy}{dx}$$

Use the product rule to differentiate $2xy$.

$$2x\frac{dy}{dx} - 2y\frac{dy}{dx} - 4\frac{dy}{dx} = 10 - 2x - 2y$$

Collect '$\frac{dy}{dx}$' terms on one side, and everything else on the other side.

$$\frac{dy}{dx} = \frac{10 - 2x - 2y}{2x - 2y - 4} = \frac{5 - x - y}{x - y - 2}$$

So when $\frac{dy}{dx} = 0$, $\frac{5 - x - y}{x - y - 2} = 0 \Rightarrow 5 - x - y = 0 \Rightarrow y = 5 - x$

This is one mammoth example, isn't it?
Let's just take a break with this llama for a minute.

Ahh, so peaceful...

Right then — back to work.

b) You can **use** the answer to part a) in the equation of the **curve** to find the points where $\frac{dy}{dx} = 0$.

When $\frac{dy}{dx} = 0$, $y = 5 - x$. So at the stationary points,

$$x^2 + 2xy - y^2 = 10x + 4y - 21$$

$$x^2 + 2x(5 - x) - (5 - x)^2 = 10x + 4(5 - x) - 21$$

Substitute $y = 5 - x$ into the original equation to find the values of x at the stationary points.

$$x^2 + 10x - 2x^2 - 25 + 10x - x^2 = 10x + 20 - 4x - 21$$

$$-2x^2 + 20x - 25 = 6x - 1$$

$$-2x^2 + 14x - 24 = 0$$

$$x^2 - 7x + 12 = 0$$

Simplify until you're left with a quadratic that you can solve.

$$(x - 3)(x - 4) = 0$$

$$x = 3 \text{ or } x = 4$$

$$x = 3 \Rightarrow y = 5 - 3 = 2 \quad \text{and} \quad x = 4 \Rightarrow y = 5 - 4 = 1$$

Don't forget to find the y-values, since the question asks you for the coordinates.

So the stationary points of A are **(3, 2)** and **(4, 1)**.

Implicit Differentiation

You can differentiate *Inverse Trig Functions* implicitly

Once upon a time, you met the inverse trig functions **arcsin**, **arccos** and **arctan** (well, actually it was on p.66). You can use implicit differentiation to differentiate these little blighters — the method goes a little bit like this:

Example: Use implicit differentiation to show that, if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Take sin of both sides to get rid of the arcsin: $\sin y = x$

Now use **implicit differentiation**: $\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

Use the **chain rule** to deal with $\frac{d}{dx}(\sin y)$: $\frac{d}{dy}(\sin y)\frac{dy}{dx} = 1$

$\cos y \frac{dy}{dx} = 1$

Rearrange to make $\frac{dy}{dx}$ the subject: $\frac{dy}{dx} = \frac{1}{\cos y}$

Use $\cos^2\theta + \sin^2\theta \equiv 1$ to get this in terms of $\sin y$: $\frac{dy}{dx} = \frac{1}{\sqrt{\cos^2 y}} = \frac{1}{\sqrt{1-\sin^2 y}}$

See p.68 for a reminder of your go-to trig identities.

Now use the equation $\sin y = x$ to get rid of y: $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ as required.

You can also differentiate arccos and arctan the same way — here are the answers you should get:

f(x) =	arcsin x	arccos x	arctan x
f′(x) =	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

Practice Questions

Q1 Use implicit differentiation to find $\frac{dy}{dx}$ for each of the following equations:

a) $4x^2 - 2y^2 = 7x^2y$ b) $3x^4 - 2xy^2 = y$ c) $(\cos x)(\sin y) = xy$

Q2 Using your answers to question 1, find:

a) the gradient of the tangent to the graph of $4x^2 - 2y^2 = 7x^2y$ at (1, –4),

b) the gradient of the normal to the graph of $3x^4 - 2xy^2 = y$ at (1, 1).

Exam Questions

Q1 The equation of curve C is $6x^2y - 7 = 5x - 4y^2 - x^2$.

a) The line T has the equation $y = c$ and passes through a point on C where $x = 2$. Find c, given that $c > 0$. [2 marks]

b) T also crosses C at point Q.

(i) Find the coordinates of Q. [2 marks]

(ii) Find the gradient of C at Q. [6 marks]

Q2 The curve C has the equation $3e^x + 6y = 2x^2y$.

a) Use implicit differentiation to find an expression for $\frac{dy}{dx}$. [3 marks]

b) Show that at the stationary points of C, $y = \frac{3e^x}{4x}$. [2 marks]

c) Hence find the exact coordinates of the two stationary points of C. [4 marks]

Q3 Use implicit differentiation to differentiate the function $f(x) = \arctan x$. [5 marks]

If an imp asks to try your ice lolly, don't let the imp lick it...

Wowzers — implicit differentiation really is a bit more involved than regular old differentiation... Well, I'm happy to inform you that this is the end of differentiation as we know it (well, as you need to know it for the exams anyway). There was a lot to get to grips with in this section, so make sure it's all sunk in before moving on to the next bit.