Differentiation with Parametric Equations

Oh look — it's your old friend, parametric equations. If you've forgotten about them already, go look them up ~~on social media~~ in Section 3. Don't look up parametric equations on social media — they always post such rubbish...

Differentiating Parametric Equations *is a lot* ***Simpler*** *than you might expect*

Just suppose you've got a **curve** defined by two **parametric equations**, with the parameter t: $y = f(t)$ and $x = g(t)$.

If you can't find the **Cartesian equation**, it seems like it would be a bit tricky to find the gradient, $\frac{dy}{dx}$.

Luckily the chain rule (see p.94) is on hand to help out:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

This is the same as on p.94, except we're using t instead of u, and we've replaced '$\times \frac{dt}{dx}$' with '$\div \frac{dx}{dt}$'.

Example: The curve C is defined by the parametric equations $x = t^2 - 1$ and $y = t^3 - 2t + 4$.

Find: a) $\frac{dy}{dx}$ in terms of t, b) the equation of the normal to the curve C when $t = -1$.

a) Start by **differentiating** the two parametric equations **with respect to t**: $\frac{dy}{dt} = 3t^2 - 2$, $\frac{dx}{dt} = 2t$

Now use the **chain rule** to combine them: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3t^2 - 2}{2t}$

To be continued...

Use the ***Gradient*** *to find* ***Tangents*** *and* ***Normals***

Of course, it's rarely as straightforward as just finding $\frac{dy}{dx}$. A lot of the time, you'll have to **use** it to find the equation of a **tangent** or **normal** (see p.85) to the parametric curve.

b) Find the coordinates of the point where $t = -1$: $y = (-1)^3 - 2(-1) + 4 = 5$, $x = (-1)^2 - 1 = 0$

So the point has coordinates **(0, 5)**.

Use the answer to a) to find the **gradient** when $t = -1$: $\frac{dy}{dx} = \frac{3(-1)^2 - 2}{2(-1)} = \frac{3-2}{-2} = -\frac{1}{2}$

So the normal to C has a gradient of: $-1 \div -\frac{1}{2} = 2$

The normal passes through (0, 5) and has a gradient of 2. Write this in the form $y = mx + c$: $y = 2x + 5$

It's often easier to use the equation $y - y_1 = m(x - x_1)$, but since the point you're using is (0, 5), you know that the y-intercept (c) is 5.

Practice Question

Q1 A curve is defined by the parametric equations $x = t^2$ and $y = 3t^3 - 4t$.

a) Find $\frac{dy}{dx}$ for this curve.

b) Find the coordinates of the stationary points of the curve.

Exam Questions

Q1 The curve C is defined by the parametric equations $x = 3\theta - \cos 3\theta$, $y = 2\sin\theta$, $-\pi \le \theta \le \pi$.

a) Show that the gradient of C at the point $(\pi + 1, \sqrt{3})$ is $\frac{1}{3}$. [6 marks]

b) Find the equation of the normal to C when $\theta = \frac{\pi}{6}$. [4 marks]

Q2 A curve, C, has parametric equations $x = t^2 + 2t - 3$, $y = 2 - t^3$.

a) The line L is the tangent to C at $y = -6$. Show that the equation of L is $y = -2x + 4$. [4 marks]

b) L also meets C at point P. Find the equation of the normal to the curve C at P. [7 marks]

You can figure out when it's going to rain using barometric differentiation...

If you ask me, having a 'find the tangent/normal' part in one of these questions is like having chocolate on a digestive biscuit — it makes it at least 4 times better (and in case you were wondering, tangent = milk and normal = dark).