Differential Equations

You might be given Extra Information

1) In the exam, you might be given a question that uses differential equations to **model** a **real-life problem**.
2) **Population** questions are an example of this — the population might be **increasing** or **decreasing**, and you have to find and solve differential equations to show it. In cases like this, one of the variables will usually be t, **time**.
3) You might be given a **starting condition** — e.g. the **initial population**. The important thing to remember is that:

The starting condition occurs when $t = 0$.

This is pretty obvious, but it's really important.

4) You might also be given **extra information** — e.g. a **specific population** (where you have to figure out what t is when the population reaches this number), or a **specific time** (where you have to work out what the population will be at this time). Make sure you always **link** the numbers you get back to the **situation**.

Example: The population of rabbits in a park is decreasing as winter approaches. The decrease in the population, P, after t days, is modelled by the differential equation $\frac{dP}{dt} = -0.1P$. Find the time at which the population of rabbits will have halved, to the nearest day.

First, solve the differential equation to find the general solution: $\frac{dP}{dt} = -0.1P \Rightarrow \frac{1}{P}\,dP = -0.1\,dt$

Integrating this gives: $\int \frac{1}{P}\,dP = \int -0.1\,dt \Rightarrow \ln P = -0.1t + C$

You don't need modulus signs for ln P as P ≥ 0 — you can't have a negative population.

At $t = 0$, $P = P_0$. Putting these values into the equation gives: $\ln P_0 = -0.1(0) + C \Rightarrow \ln P_0 = C$

So the differential equation becomes: $\ln P = -0.1t + \ln P_0 \Rightarrow P = e^{(-0.1t + \ln P_0)} = e^{-0.1t}e^{\ln P_0} \Rightarrow P = P_0e^{-0.1t}$

Remember that $e^{\ln x} = x = \ln e^x$.

When the population of rabbits has halved, $P = \frac{1}{2}P_0$:

$$\frac{1}{2}P_0 = P_0e^{-0.1t} \Rightarrow \frac{1}{2} = e^{-0.1t} \Rightarrow \ln\frac{1}{2} = -0.1t \Rightarrow -0.6931... = -0.1t \Rightarrow t = 6.931...$$

So, to the nearest day, it will take **7 days** for the population of rabbits to halve.

You could also be asked to talk about **limitations** of a model, and suggest possible **changes** that would **improve** it. Think about things like:

- **missing information** (e.g. **above**, you weren't told P_0),
- what happens for really **big/small** values of the variables (e.g. as t gets large, P gets small but **never** reaches 0),
- how **appropriate** the model is (e.g. $P_0e^{-0.1t}$ is a **continuous** function, but population is a **discrete** variable),
- **any other factors** that haven't been included (e.g. what happens to the poor rabbits when **winter** arrives).

Practice Question

Q1 Find the general solution to the following differential equations, giving your answers in the form $y = f(x)$:

a) $\frac{dy}{dx} = \frac{1}{y}\cos x \quad (y > 0)$ b) $\frac{dy}{dx} = 2y^2 - 3(xy)^2$ c) $\frac{dy}{dx} = e^{x-y}$

Exam Questions

Q1 a) Find the general solution to the differential equation $\frac{dy}{dx} = \frac{\cos x \cos^2 y}{\sin x}$, $0 < x < \pi$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$. [4 marks]

b) Given that $y = 0$ when $x = \frac{\pi}{6}$, solve the differential equation above. [2 marks]

Q2 A company sets up an advertising campaign to increase sales of margarine. After the campaign, the number of tubs of margarine sold each week, m, increases over time, t (in weeks), at a rate that is directly proportional to the square root of the number of tubs sold.

a) Formulate a differential equation in terms of t, m and a constant k. [2 marks]

b) At the start of the campaign, the company was selling 900 tubs of margarine a week. Use this information to solve the differential equation, giving m in terms of k and t. [5 marks]

c) Hence calculate the number of tubs sold in the fifth week after the campaign, given that $k = 2$. [2 marks]

d) Explain why the model is not likely to be accurate for large values of t. [2 marks]

At t = 10, we kill all the bunnies...

These questions can get a bit morbid — just how I like them. They might look a bit scary, as they throw a lot of information at you in one go, but once you know how to solve them, they're a walk in the park. Rabbit traps optional.