Integrating e^x and $1/x$

Sometimes you get integrals that look really nasty — like exponentials and fractions.
However, once you learn the clever tricks for dealing with them, they're quite easy to integrate really.

e^x integrates to give e^x (+ C)

As e^x **differentiates** to give e^x (see p.96), it makes sense that

$$\int e^x\,dx = e^x + C$$

Don't forget the constant of integration.

Once you're happy with that, you can use it to solve lots of integrations that have an e^x term in them.
If the **coefficient** of x isn't 1, you need to **divide** by that coefficient when you **integrate** — so $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

Examples: Integrate the following: a) e^{7x} b) $2e^{4-3x}$ c) $e^{\frac{x}{2}}$

a) $\int e^{7x}dx = \frac{1}{7}e^{7x} + C$

If you differentiated e^{7x} using the chain rule, you'd get $7e^{7x}$. So when you integrate, you need to **divide by 7** (the coefficient of x). This is so that if you differentiated your answer you'd get back to e^{7x}.

b) $\int 2e^{4-3x}dx = -\frac{2}{3}e^{4-3x} + C$

This one isn't as bad as it looks — if you differentiated $2e^{4-3x}$, you'd get $-6e^{4-3x}$, so you need to **divide by –3** (the coefficient of x) when you integrate. Differentiating your answer gives you $2e^{4-3x}$.

c) $\int e^{\frac{x}{2}}dx = \int e^{\frac{1}{2}x}dx = 2e^{\frac{x}{2}} + C$

If you differentiated $e^{\frac{x}{2}}$ using the chain rule, you'd get $\frac{1}{2}e^{\frac{x}{2}}$, so you need to **multiply by 2** when you integrate.

You can always differentiate your answer to check that it works.

$1/x$ integrates to $\ln|x|$ (+ C)

When you started integration on page 108, you couldn't integrate $\frac{1}{x}$ ($= x^{-1}$) by **increasing** the **power** by 1 and **dividing** by it, as you ended up **dividing by 0** (which is baaaaad).
However, on p.96, you saw that $\ln x$ **differentiates** to give $\frac{1}{x}$, so:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Don't worry about where the modulus sign (see p.28) comes from — using $|x|$ just means that there isn't a problem when x is negative.

There's also a general version of this rule:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

Examples: Integrate the following: a) $\frac{5}{x}$ b) $\frac{1}{3x}$ c) $\frac{1}{4x+5}$

a) $\int \frac{5}{x}dx = 5\int \frac{1}{x}dx = 5\ln|x| + C$

5 is a constant coefficient — you can take it outside the integral if you want.

You could also write $5\ln|x|$ as $\ln|x^5|$.

b) $\int \frac{1}{3x}dx = \frac{1}{3}\int \frac{1}{x}dx = \frac{1}{3}\ln|x| + C$

Don't make the mistake of putting just $\ln|3x| + C$, as this would differentiate to give $\frac{1}{x}$ ($\ln|3x| = \ln|x| + \ln 3$, so $\ln 3$ disappears when you differentiate).
If you use the general rule here, you get $\frac{1}{3}\ln|3x| + C$. The two answers are the same because $\frac{1}{3}\ln|3x| = \frac{1}{3}\ln|x| + \frac{1}{3}\ln 3$, so it's just the value of C that's different.

c) $\int \frac{1}{4x+5}dx = \frac{1}{4}\ln|4x+5| + C$

Use the general rule for this one — don't forget to divide by a (in this case, 4).

If you have a fraction that has a **function of x** in the **numerator** and a **different function of x** in the **denominator** (e.g. $\frac{x-2}{x^3+1}$), you won't be able to use this formula to integrate it.
However, if you can **differentiate** the **denominator** to get the **numerator** (e.g. $\frac{3x^2}{x^3+1}$, since the derivative of $x^3 + 1$ is $3x^2$), then there's a useful trick that you can use instead...

Integrating e^x and 1/x

*Some **Fractions** integrate to **ln***

In general terms, if you're trying to integrate a fraction where the **numerator** is the **derivative** of the **denominator**, you can use this formula:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

This is another one that comes from the chain rule (p.94) — if you differentiated ln |f(x)|, you'd end up with the fraction on the left.

The hardest bit about questions like this is **recognising** that the denominator **differentiates** to give the numerator. Once you've spotted that, it's dead easy.

The numerator might be a **multiple** of the derivative of the denominator, just to confuse things, so watch out for that.

Examples: Find: a) $\int \frac{8x^3-4}{x^4-2x}\,dx$, b) $\int \frac{3\sin 3x}{\cos 3x+2}\,dx$.

a) $\frac{d}{dx}(x^4 - 2x) = 4x^3 - 2$
and $8x^3 - 4 = 2(4x^3 - 2)$
The numerator is 2 × the derivative of the denominator, so:

$$\int \frac{8x^3-4}{x^4-2x}\,dx = 2\int \frac{4x^3-2}{x^4-2x}\,dx$$
$$= \mathbf{2\ln|x^4-2x| + C}$$

b) $\frac{d}{dx}(\cos 3x + 2) = -3\sin 3x$
The numerator is −1 × the derivative of the denominator, so:

$$\int \frac{3\sin 3x}{\cos 3x+2}\,dx = -\int \frac{-3\sin 3x}{\cos 3x+2}\,dx$$
$$= \mathbf{-\ln|\cos 3x+2| + C}$$

Using C = −ln k, you could combine all the terms into one using the laws of logs: $-\ln|\cos 3x + 2| - \ln k = -\ln|k(\cos 3x + 2)|$

*You can use **Partial Fractions** to integrate*

In Section 2 (pages 14-15), you saw how to **break down** a scary-looking **algebraic fraction** into **partial fractions**. This comes in pretty handy when you're **integrating —** you could try other methods, but it would get **messy** and probably **end in tears**. Fortunately, once you've split it up into **partial fractions**, it's much **easier** to integrate.

Example: Find $\int \frac{12x+6}{4x^2-9}\,dx$.

This is the example from p.14, and it can be written as partial fractions like this: $\frac{2}{2x+3} + \frac{4}{2x-3}$

Integrating the partial fractions is much easier: $\int \left(\frac{2}{2x+3} + \frac{4}{2x-3}\right)dx = \ln|2x+3| + 2\ln|2x-3| + C$
$= \mathbf{\ln|(2x+3)(2x-3)^2| + C}$

Be careful with the coefficients here. Try differentiating them to see where they come from if you're not sure.

This answer's been simplified using the log laws (see p.76).

Practice Questions

Q1 Find: a) $\int 4e^{2x}\,dx$, b) $\int e^{3x-5}\,dx$, c) $\int \frac{2}{3x}\,dx$, d) $\int \frac{2}{2x+1}\,dx$.

Q2 Integrate $\int \frac{20x^4+12x^2-12}{x^5+x^3-3x}\,dx$.

Q3 $\frac{3x+10}{(2x+3)(x-4)} \equiv \frac{A}{2x+3} + \frac{B}{x-4}$. Find the value of A and B, and hence find $\int \frac{3x+10}{(2x+3)(x-4)}\,dx$.

Exam Questions

Q1 Find $\int 3e^{(5-6x)}\,dx$. [2 marks]

Q2 Given that $x = 1$ is a root of $x^3 - 6x^2 + 11x - 6$, completely factorise $x^3 - 6x^2 + 11x - 6$, and hence find $\int \frac{4x-10}{x^3-6x^2+11x-6}\,dx$. [8 marks]

Don't cheat in your exams — copying's derivative, so revising's integral...

There's quite a lot of different techniques to take in on these pages. Don't forget to try differentiating your answer at the end, so you know if it's gone wrong somewhere — checking my answers has saved my skin more times than I can tell you (28). Make sure you're a master of partial fractions before you even think about integrating them.

Integrating Trig Functions

You thought you'd killed off the dragon that is trigonometry back in Section 5, but it rears its ugly head again now — you need to know how to integrate trig functions. Equip your most trusty dragon-slaying sword and read on...

Sin and *Cos* are *Easy* to integrate

On page 98, you saw that **sin x** differentiates to give **cos x**, **cos x** differentiates to give **–sin x** and **tan x** differentiates to give **sec²x** (where the angle x is in **radians**). So it's pretty obvious that:

$$\int \sin x \, dx = -\cos x + C$$
$$\int \cos x \, dx = \sin x + C$$
$$\int \sec^2 x \, dx = \tan x + C$$

Integrating tan x is a bit different — see the next page.

The derivative of tan x is on the formula sheet, so you can work backwards to this result.

If x has a **coefficient** that **isn't 1** (e.g. sin $3x$), you just **divide** by the **coefficient** when you integrate — just like on page 114.

Example: Find $\int \left(\cos 4x - 2\sin 2x + \sec^2 \frac{1}{2}x\right) dx$.

Integrate each term separately using the results from above:

$\int \cos 4x \, dx = \frac{1}{4}\sin 4x \qquad \int -2\sin 2x \, dx = -2\left(-\frac{1}{2}\cos 2x\right) = \cos 2x \qquad \int \sec^2 \frac{1}{2}x \, dx = \frac{1}{\frac{1}{2}}\tan \frac{1}{2}x = 2\tan \frac{1}{2}x$

Putting these terms together and adding the constant gives:

$$\int \left(\cos 4x - 2\sin 2x + \sec^2 \frac{1}{2}x\right) dx = \frac{1}{4}\sin 4x + \cos 2x + 2\tan \frac{1}{2}x + C$$

There are some *Results* you can just *Learn*

There's a list of **trig integrals** that you can just **learn** — you don't need to know where they came from, you can just **use** them. You've met these ones before — they're the **results** of differentiating **cosec x**, **sec x** and **cot x**.

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$
$$\int \sec x \tan x \, dx = \sec x + C$$
$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

You saw how to differentiate cosec, sec and cot on p.102.

The coefficients of x have to be the same in each term — e.g. you couldn't integrate sec x tan 3x using this method.

As usual, you need to **divide** by the **coefficient of x** when you integrate.

Example: Find $\int \left(10\sec 5x \tan 5x + \frac{1}{2}\operatorname{cosec} 3x \cot 3x - \operatorname{cosec}^2(6x+1)\right) dx$.

This one looks a bit scary, but take it one step at a time. Integrate each bit in turn to get:

$$\int 10 \sec 5x \tan 5x \, dx = 10\left(\frac{1}{5}\sec 5x\right) = 2\sec 5x$$
$$\int \frac{1}{2}\operatorname{cosec} 3x \cot 3x \, dx = \frac{1}{2}\left(-\frac{1}{3}\operatorname{cosec} 3x\right) = -\frac{1}{6}\operatorname{cosec} 3x$$
$$\int -\operatorname{cosec}^2(6x+1) \, dx = -\left(-\frac{1}{6}\cot(6x+1)\right) = \frac{1}{6}\cot(6x+1)$$

The + 1 inside the brackets has no effect on the integration — differentiate to see why.

Putting these terms together and adding the constant gives:

$$\int \left(10\sec 5x \tan 5x + \frac{1}{2}\operatorname{cosec} 3x \cot 3x - \operatorname{cosec}^2(6x+1)\right) dx = 2\sec 5x - \frac{1}{6}\operatorname{cosec} 3x + \frac{1}{6}\cot(6x+1) + C$$

Integrating Trig Functions

You can get ln of Trig Functions too

On page 115 you saw that there was a **formula** for integrating a **fraction** where the **numerator** was the **derivative** of the **denominator** (or a **multiple** of it anyway). You may also have noticed that we've very suspiciously not integrated tan x yet. Well, we can use our fraction trick to finally work out the integral of tan:

$\tan x = \frac{\sin x}{\cos x}$ and $\frac{d}{dx}(\cos x) = -\sin x$, so you can write $\int \tan x\, dx$ in the form $\int -\frac{f'(x)}{f(x)}\, dx$:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\ln|\cos x| + C$$

$-\ln|\cos x|$ is the same as $\ln|\sec x|$ — this comes from the laws of logs (see p.76).

You can integrate some other **trig functions** in a similar way:

Example: Show that $\int \cot x\, dx = \ln|\sin x| + C$.

Use a similar method to the box above:

- Write cot x as a fraction: $\cot x = \frac{\cos x}{\sin x}$
- Differentiate the denominator: $\frac{d}{dx}(\sin x) = \cos x$
- Use the formula: $\int \cot x\, dx = \int \frac{f'(x)}{f(x)}\, dx$, where $f(x) = \sin x$

So $\int \cot x\, dx = \ln|f(x)| + C$

$= \ln|\sin x| + C$ as required

The Double Angle Formulas are useful for Integration

If you're given a tricky **trig function** to integrate, see if you can **simplify** it using one of the **double angle formulas**. They're especially useful for things like $\cos^2 x$, $\sin^2 x$ and $\sin x \cos x$. Here are the double angle formulas (see p.71):

$$\sin 2x \equiv 2 \sin x \cos x \qquad \cos 2x \equiv \cos^2 x - \sin^2 x$$

$$\tan 2x \equiv \frac{2\tan x}{1 - \tan^2 x} \qquad \cos 2x \equiv 2\cos^2 x - 1 \qquad \cos 2x \equiv 1 - 2\sin^2 x$$

You can rearrange the second two cos 2x formulas to get expressions for $\cos^2 x$ and $\sin^2 x$:

$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Once you've **replaced** the **original function** with one of the **double angle formulas**, you can just **integrate** as normal.

Examples: Find: a) $\int \sin^2 x\, dx$, b) $\int \cos^2 5x\, dx$, c) $\int \sin x \cos x\, dx$.

a) Using the double angle formula above, write $\sin^2 x$ as $\frac{1}{2}(1 - \cos 2x)$, then integrate.

$$\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

b) Using the double angle formula above, write $\cos^2 5x$ as $\frac{1}{2}(\cos 10x + 1)$, then integrate.

$$\int \cos^2 5x\, dx = \int \frac{1}{2}(\cos 10x + 1) = \frac{1}{2}\left(\frac{1}{10}\sin 10x + x\right) + C = \frac{1}{20}\sin 10x + \frac{1}{2}x + C$$

Don't forget to double the coefficient of x here. You'll also need to divide by 10 when you integrate.

c) Using the double angle formula above, write $\sin x \cos x$ as $\frac{1}{2}\sin 2x$, then integrate.

$$\int \sin x \cos x\, dx = \int \frac{1}{2}\sin 2x\, dx = \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + C = -\frac{1}{4}\cos 2x + C$$

Integrating Trig Functions

*Use the **Identities** to get a function you **Know** how to **Integrate***

There are a couple of other **identities** you can use to **simplify trig functions** (see p.68):

$$\sec^2\theta \equiv 1 + \tan^2\theta \qquad\qquad \text{cosec}^2\theta \equiv 1 + \cot^2\theta$$

These two identities are really useful if you have to integrate $\tan^2 x$ or $\cot^2 x$, as you already know how to integrate $\sec^2 x$ and $\text{cosec}^2 x$ (see p.116). Don't forget the stray **1s** flying around — they'll just integrate to x.

Examples: Find: a) $\int(\tan^2 x - 1)\,dx$, b) $\int(\cot^2 3x)\,dx$.

a) Rewrite the function in terms of $\sec^2 x$:
$\tan^2 x - 1 \equiv \sec^2 x - 1 - 1 \equiv \sec^2 x - 2$.
Now integrate:
$\int(\sec^2 x - 2)\,dx = \tan x - 2x + C$

b) Get the function in terms of cosec^2:
$\cot^2 3x \equiv \text{cosec}^2 3x - 1$.
Now integrate:
$\int(\text{cosec}^2 3x - 1)\,dx = -\frac{1}{3}\cot 3x - x + C$

Remember to divide by 3, the coefficient of x, when you integrate.

Example: Evaluate $\int_0^{\frac{\pi}{3}}\left(6\sin 3x\cos 3x + \tan^2\frac{1}{2}x + 1\right)dx$.

Using the identities, $6\sin 3x\cos 3x \equiv 3\sin 6x$
and $\tan^2\frac{1}{2}x + 1 \equiv \sec^2\frac{1}{2}x$ gives:

$$\int_0^{\frac{\pi}{3}}\left(3\sin 6x + \sec^2\frac{1}{2}x\right)dx = \left[-\frac{3}{6}\cos 6x + 2\tan\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$$

$$= \left[-\frac{1}{2}\cos 6\left(\frac{\pi}{3}\right) + 2\tan\frac{1}{2}\left(\frac{\pi}{3}\right)\right] - \left[-\frac{1}{2}\cos 6(0) + 2\tan\frac{1}{2}(0)\right]$$

$$= \left[-\frac{1}{2}\cos(2\pi) + 2\tan\left(\frac{\pi}{6}\right)\right] - \left[-\frac{1}{2}\cos(0) + 2\tan(0)\right]$$

$$= \left[-\frac{1}{2}(1) + 2\left(\frac{1}{\sqrt{3}}\right)\right] - \left[-\frac{1}{2}(1) + 2(0)\right] = -\frac{1}{2} + \frac{2}{\sqrt{3}} + \frac{1}{2} = \frac{2}{\sqrt{3}}$$

Use the table of common trig angles on p.56 to help you here.

Sadly, Boris' identity wasn't making it any easier to integrate at all.

Practice Questions

Q1 Find $\int(\cos 4x - \sec^2 7x)\,dx$.

Q2 Evaluate $\int\left(6\sec 3x\tan 3x - \text{cosec}^2\frac{x}{5}\right)dx$.

Q3 Integrate $\int -\frac{\cos x}{3\sin x}\,dx$.

Q4 Use an appropriate trig identity to find $\int\frac{2\tan 3x}{1-\tan^2 3x}\,dx$.

Q5 Use the trig identity $\sec^2 x \equiv 1 + \tan^2 x$ to find $\int(2\tan^2 3x + 2)\,dx$.

Exam Questions

Q1 Find $\int\frac{\text{cosec}^2 x - 2}{\cot x + 2x}\,dx$. [3 marks]

Q2 Use an appropriate identity to find $\int 2\cot^2 x\,dx$. [3 marks]

This is startin' to grate on me now...

Some of these integrals are on the formula sheet. The ones from the bottom of p.116 are on the list of differentiation formulas, so if you can work backwards (i.e. from f'(x) to f(x)), you can sneakily use these results without having to remember them at all. Don't tell the examiners though — I don't think they've noticed yet... *(disclaimer: they definitely have)*