Further Definite Integrals

What's that? You want some more tasty examples of definite integrals, with parametric equations thrown in to spice things up a bit? It's your lucky day, my friend — read on.

*Sometimes you have to **Subtract** integrals*

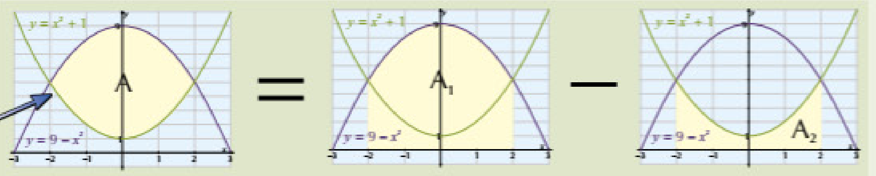
As on page 111, it's best to draw a **sketch** to work out exactly what you need to do.

Example: Find the area enclosed by the curves $y = x^2 + 1$ and $y = 9 - x^2$.

Solve $x^2 + 1 = 9 - x^2$ to find where the curves meet:

$$x^2 + 1 = 9 - x^2 \Rightarrow 2x^2 = 8$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

So you'll have to integrate between −2 and 2.



The area under the purple curve is:

$$A_1 = \int_{-2}^{2} (9 - x^2)\,dx$$
$$= \left[9x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= \left(18 - \frac{2^3}{3}\right) - \left(-18 - \frac{(-2)^3}{3}\right)$$
$$= \left(18 - \frac{8}{3}\right) - \left(-18 - \frac{-8}{3}\right)$$
$$= \frac{46}{3} - \left(-\frac{46}{3}\right) = \frac{92}{3}$$

The area under the green curve is:

$$A_2 = \int_{-2}^{2} (x^2 + 1)\,dx$$
$$= \left[\frac{x^3}{3} + x\right]_{-2}^{2}$$
$$= \left(\frac{2^3}{3} + 2\right) - \left(\frac{(-2)^3}{3} + (-2)\right)$$
$$= \left(\frac{8}{3} + 2\right) - \left(\frac{-8}{3} - 2\right)$$
$$= \frac{14}{3} - \left(-\frac{14}{3}\right) = \frac{28}{3}$$

And the area you need is the difference between these:

$$A = A_1 - A_2$$
$$= \frac{92}{3} - \frac{28}{3} = \frac{64}{3}$$

Instead of integrating before subtracting — you could try 'subtracting the curves', and then integrating. This overall area is also:
$$A = \int_{-2}^{2} [(9 - x^2) - (x^2 + 1)]\,dx$$

*Use the **Chain Rule** to **Integrate Parametrics***

1) Normally, to find the area under a graph, you can do a simple integration. But if you've got **parametric equations** (see page 40), things are more difficult — you can't find $\int y\,dx$ if y isn't written in terms of x.

2) There's a sneaky way to get around this. Suppose your parameter's t, then

$$\int y\,dx = \int y\frac{dx}{dt}\,dt$$

3) Both y and $\frac{dx}{dt}$ are written in terms of t, so you can **multiply** them together to get an expression you can **integrate** with respect to t.

This comes from the chain rule (see p.94) — if you think of dx as $\frac{dx}{1}$, then $\frac{dx}{1} = \frac{dx}{dt} \times \frac{dt}{1}$.

Example: a) A curve is defined by the parametric equations $x = t^3 + 3$ and $y = t^2 + 2t + 3$. Show that $\int y\,dx = \int (3t^4 + 6t^3 + 9t^2)\,dt$.

$\frac{dx}{dt} = 3t^2$, so using the formula above:

$$\int y\,dx = \int y\frac{dx}{dt}\,dt = \int (t^2 + 2t + 3)(3t^2)\,dt = \int (3t^4 + 6t^3 + 9t^2)\,dt.$$

b) Hence, find $\int y\,dx$ between $t = 0$ and $t = 1$.

You know that $\int y\,dx = \int (3t^4 + 6t^3 + 9t^2)\,dt$, so integrate between the given limits:

$$\int_0^1 3t^4 + 6t^3 + 9t^2\,dt = \left[\frac{3t^5}{5} + \frac{3t^4}{2} + 3t^3\right]_0^1$$
$$= \left[\frac{3(1^5)}{5} + \frac{3(1^4)}{2} + 3(1^3)\right] - [0 + 0 + 0]$$
$$= 0.6 + 1.5 + 3 = 5.1$$

These terms have been simplified from $\frac{6t^4}{4}$ and $\frac{9t^3}{3}$ respectively.

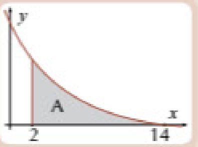
Asparametric equations are integral to any summer dish.

Further Definite Integrals

*Remember to **Convert** the **Limits** of **Definite Parametric Integrals***

With a **definite integral** involving parametric equations, you need to **alter the limits** as well. You can do this by **substituting** the x-limits into the parametric equation for x to find the corresponding values of t.

Example: The shaded region marked A on this sketch is bounded by the lines $y = 0$ and $x = 2$, and by the curve with parametric equations $x = t^2 - 2$ and $y = t^2 - 9t + 20$, $t \geq 0$, which crosses the x-axis at $x = 14$.
Find the area of A.



You need to use $\int y\,dx = \int y \frac{dx}{dt}\,dt$, so find: $\frac{dx}{dt} = \frac{d}{dt}(t^2 - 2) = 2t$

Now you need to convert the **limits**.
2 and 14 are the limits **with respect to x**,
so find the **corresponding values of t**:

$x = 2 \Rightarrow t^2 - 2 = 2 \Rightarrow t^2 = 4 \Rightarrow t = 2$
$x = 14 \Rightarrow t^2 - 2 = 14 \Rightarrow t^2 = 16 \Rightarrow t = 4$

You can ignore the negative solutions here as you're told $t \geq 0$ in the question.

Now **integrate** to find the area of A:

$$A = \int_2^{14} y\,dx = \int_2^4 y\frac{dx}{dt}\,dt = \int_2^4 (t^2 - 9t + 20)(2t)\,dt = \int_2^4 (2t^3 - 18t^2 + 40t)\,dt$$

$$= \left[\frac{1}{2}t^4 - 6t^3 + 20t^2\right]_2^4 = \left(\frac{1}{2}(4)^4 - 6(4)^3 + 20(4)^2\right) - \left(\frac{1}{2}(2)^4 - 6(2)^3 + 20(2)^2\right) = 64 - 40 = 24$$

Practice Questions

Q1 Find the area bounded by the curve $y = 1 - x^2$ and the line $y = 1 - x$.

Q2 A curve has parametric equations $y = 4 + \frac{3}{t}$ and $x = t^2 - 1$. Write the integral $\int y\,dx$ in the form $\int f(t)\,dt$.

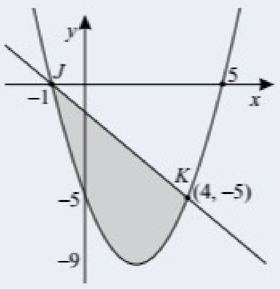
Exam Questions

Q1 The diagram on the right shows the curve $y = (x + 1)(x - 5)$.
Points J $(-1, 0)$ and K $(4, -5)$ lie on the curve.

a) Find the equation of the straight line joining J and K in the form $y = mx + c$. [2 marks]

b) Calculate $\int_{-1}^{4} (x + 1)(x - 5)\,dx$. [5 marks]

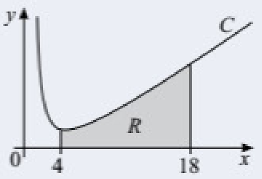
c) Find the area of the shaded region. [3 marks]



Q2 The curve C has parametric equations $x = t^2 + 3t$ and $y = t^2 + \frac{1}{t^3}$, $t > 0$.
The shaded region marked R is enclosed by C, the x-axis and the lines $x = 4$ and $x = 18$.

a) Show that the area of R is given by $\int_1^3 \frac{(t^5 + 1)(2t + 3)}{t^3}\,dt$. [4 marks]

b) Hence, find the area of R. [4 marks]

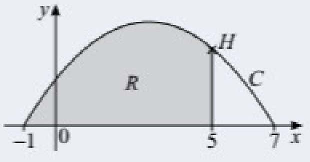


Q3 The parametric equations of curve C are $x = 3 + 4\sin\theta$, $y = \frac{1 + \cos 2\theta}{3}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
Point H on C has coordinates $\left(5, \frac{1}{2}\right)$.

a) Find the value of θ at point H. [2 marks]

The region R is enclosed by C, the line $x = 5$, and the x-axis, as shown.

b) Show that the area of R is given by the integral $\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^3\theta\,d\theta$. [5 marks]



Sick of regular integration? Call the parametrics...

If you're asked to find an area, it's really useful to draw a sketch (or annotate, if given in the question). To integrate a curve defined by parametric equations, you'll need to remember the formula on page 112 — or know the trick to derive it using the chain rule. The limits will need converting to the corresponding values of t (or whatever) as well.