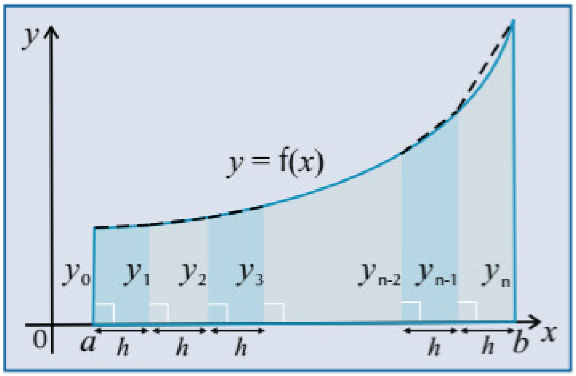
Numerical Integration

Let's leave roots behind for now and have a look at numerical methods for integration. Sometimes the algebraic methods you saw in Section 8 just won't get the job done. That's when the trapezium rule comes in handy...

*The **Trapezium Rule** is used to find the **Approximate Area** under a curve*



Instead of using **integration** to find the **exact area** under a curve (see p.110), you can find an **approximation** by summing the areas of **trapeziums** drawn between the limits a and b. This is known as the **trapezium rule**.

The area of each trapezium is $A = \frac{h}{2}(y_n + y_{n+1})$.

The area represented by $\int_a^b y\,dx$ is approximately:

$$\int_a^b y\,dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$$

where $\boldsymbol{n}$ is the number of strips or intervals and $\boldsymbol{h}$ is the width of each strip.

This will be on the formula sheet in the exam — but you'll need to remember how to use it.

You can find the width of each strip using $\boldsymbol{h = \frac{(b-a)}{n}}$.

$\boldsymbol{y_0, y_1, \dots, y_n}$ are the heights of the sides of the trapeziums — you get these by putting the x-values into the equation of the curve.

Example: Find an approximate value for $\int_0^2 \sqrt{4-x^2}\,dx$ using 4 strips. Give your answer to 4 s.f.

1) Start by working out the width of each strip: $h = \frac{(b-a)}{n} = \frac{(2-0)}{4} = \mathbf{0.5}$

The question specifies 4 strips, so n = 4.

2) This means the **x-values** are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$ and $x_4 = 2$.

3) Set up a **table** and work out the **y-values** (or heights) using the equation in the integral:

x	$y = \sqrt{4-x^2}$
$x_0 = 0$	$y_0 = \sqrt{4-0^2} = \mathbf{2}$
$x_1 = 0.5$	$y_1 = \sqrt{4-0.5^2} = \mathbf{\sqrt{3.75}}$
$x_2 = 1.0$	$y_2 = \sqrt{4-1^2} = \mathbf{\sqrt{3}}$
$x_3 = 1.5$	$y_3 = \sqrt{4-1.5^2} = \mathbf{\sqrt{1.75}}$
$x_4 = 2.0$	$y_4 = \sqrt{4-2^2} = \mathbf{0}$

4) Now put all the y-values into the **formula** with h and n:

$$\int_0^2 \sqrt{4-x^2}\,dx$$

$$\approx \frac{0.5}{2}[2 + 2(\sqrt{3.75} + \sqrt{3} + \sqrt{1.75}) + 0]$$

$$= 0.25[2 + 2 \times 4.9914...] = 2.9957...$$

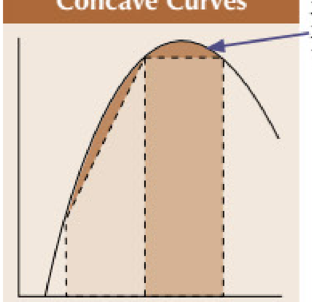
$$= \mathbf{2.996} \text{ (4 s.f.)}$$

Watch out — if they ask you to work out a question with 5 y-values (or '**ordinates**') then this is the **same** as 4 strips. The x-values usually go up in **nice jumps** — if they don't then **check** your calculations carefully.

*The **Approximation** might be an **Overestimate** or an **Underestimate***

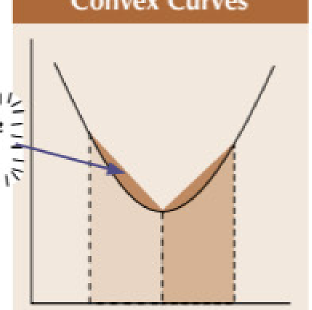
Whether the approximation is **too big** or **too small** depends on the **shape** of the curve — a sketch can show whether the tops of the trapeziums lie **above** the curve or stay **below** it.

Concave Curves

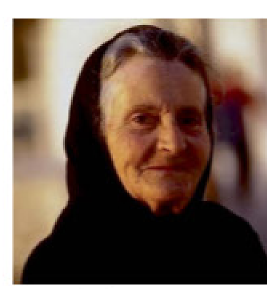


The estimate is **less** than the real area.

Convex Curves



The estimate is **more** than the real area.



Underestimate Mabel at your peril...

(See p.90 for the definitions of **concave** and **convex**.)

Numerical Integration

Use More Strips to get a More Accurate answer

Using **more strips** (i.e. increasing n) gives you a more **accurate approximation**.

Example: Use the trapezium rule to approximate the area of $\int_0^4 \frac{6x^2}{x^3+2}\,dx$, to 3 d.p., using a) $n = 2$ and b) $n = 4$.

a) For 2 strips, the width of each strip is $h = \frac{4-0}{2} = 2$, so the x-values are 0, 2 and 4.

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 2$	$y_1 = 2.4$
$x_2 = 4$	$y_2 = 1.454...$

Putting these y-values into the formula gives:

$$\int_0^4 \frac{6x^2}{x^3+2}\,dx \approx \frac{2}{2}[0 + 2(2.4) + 1.454...] = [4.8 + 1.454...] = 6.2545... = \mathbf{6.255} \text{ (3 d.p.)}$$

b) For 4 strips, the width of each strip is $h = \frac{4-0}{4} = 1$, so the x-values are 0, 1, 2, 3 and 4.

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

Putting these y-values into the formula gives:

$$\int_0^4 \frac{6x^2}{x^3+2}\,dx \approx \frac{1}{2}[0 + 2(2 + 2.4 + 1.862...) + 1.454...] = \frac{1}{2}[12.524... + 1.454...] = 6.9893... = \mathbf{6.989} \text{ (3 d.p.)}$$

If you know the exact answer, you can check how **accurate** your estimate is by working out the **percentage error**.

Example: Calculate the percentage error for a) and b) above to 2 d.p.

This was calculated using the method on p.115.

First, work out the exact value of the integral: $\int_0^4 \frac{6x^2}{x^3+2}\,dx = [2\ln|x^3+2|]_0^4 = [2\ln 66] - [2\ln 2] = 6.99301...$

For part a), the percentage error is $\frac{6.993... - 6.254...}{6.993...} \times 100 = \mathbf{10.56\%}$ (2 d.p.)

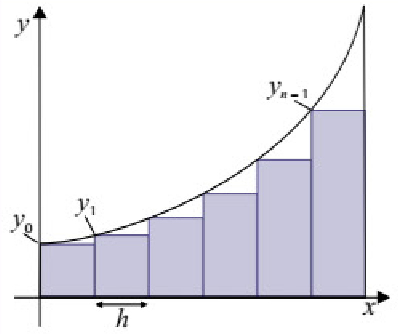
For part b) it's $\frac{6.993... - 6.989...}{6.993...} \times 100 = \mathbf{0.05\%}$ (2 d.p.)

Percentage error = $\frac{\text{actual answer} - \text{estimate}}{\text{actual answer}} \times 100$

The approximation with **more strips** has a **smaller percentage error** — so it's a **more accurate** approximation.

Find Upper and Lower Bounds using Rectangles

When you approximate the area under a curve using the trapezium rule, the approximate value will lie between **two bounds**. These bounds can be found by considering **rectangular strips** that lie **above** and **below** the curve.



One bound can be found by summing the areas of the rectangles which meet f(x) with their **left hand corner**, using this formula:

$$\int_a^b f(x)\,dx \approx h[y_0 + y_1 + y_2 + ... + y_{n-1}]$$

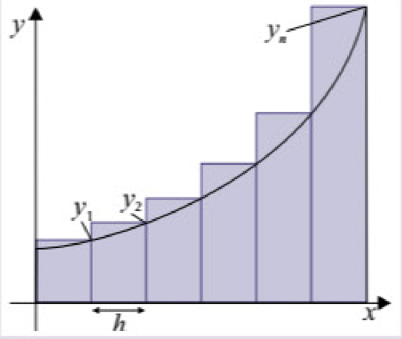
In this example, the rectangles are **below the curve**, so they give a **lower bound**.

For an **increasing function** like the one shown here, using the 'left hand corner' will give the lower bound, and using the 'right hand corner' will give the upper bound. But for a **decreasing** function, it's the other way around.

The other bound is the sum of the areas of the rectangles which meet f(x) with their **right hand corner**, using this formula:

$$\int_a^b f(x)\,dx \approx h[y_1 + y_2 + y_3 + ... + y_n]$$

Here the rectangles are **above the curve**, so they give an **upper bound**.



To calculate a bound of a curve with a turning point, do a separate calculation either side of the turning point — one will use the right hand corners, and the other will use the left.

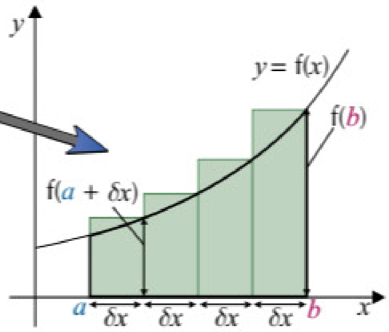
Numerical Integration

Integration is the Limit of the Sum of Rectangles

'**Differentiating from first principles**' involves finding the **gradient** of a **straight line** between two points on a curve over a **tiny interval**, δx. As the interval **approaches zero** ($\delta x \to 0$), the gradient of the line matches that of the curve. You'll be pleased to know* that you can do a similar thing for the **integral** of a curve **between two points**, a and b, using those rectangles on the previous page:

The **area** of each rectangle under the curve is its **height**, $f(x)$ × **width**, δx.
So the approximate area under the curve is the **sum** of the areas of the rectangles, from $[f(a + \delta x) \times \delta x]$ to $[f(b) \times \delta x]$.

This is written as: $$\sum_{x=a+\delta x}^{b} f(x)\delta x$$



Using more, narrower strips gives a better approximation, so as δx gets smaller and **approaches 0**, the sum matches the actual area under the curve. This is how we define an **integral** — as the **limit of a sum** of areas.

So as $\delta x \to 0$, $a + \delta x \to a$, and the lower value of the sum can be replaced with a:

$$\lim_{\delta x \to 0} \sum_{a+\delta x}^{b} f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

To write the exact integral, use d instead of δ and replace $\sum$ with $\int$ (which are both just ways of writing 'S', for 'sum').

Practice Questions

Q1 Use the trapezium rule to estimate the value of $\int_0^6 (6x - 12)(x^2 - 4x + 3)^2 \, \mathrm{d}x$, first using 4 strips and then again with 6 strips. Calculate the percentage error for each answer.

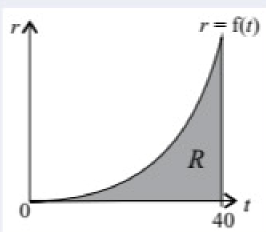
Q2 Would the trapezium rule underestimate or overestimate the area under the curve $y = e^x$? Explain why.

Q3 Find the exact upper and lower bounds of a trapezium rule approximation to $\int_0^{2\pi} \sin\left(\frac{x}{2}\right) \mathrm{d}x$ using 5 ordinates.

Exam Questions

Q1 The following experimental data is being modelled as an unknown function, $r = f(t)$, as sketched on the graph:

Time (t s)	0	5	10	15	20	25	30	35	40
Flow rate (r gs^{-1})	0	0.6	1.7	3.5	6.4	11.2	19.1	32.1	53.6



a) Use the trapezium rule with 4 strips to estimate the area under the curve, R. [3 marks]

b) Explain whether this estimate is higher or lower than the actual value of R. [1 mark]

c) Explain how a more accurate estimate could be obtained using the data available. [1 mark]

Q2 Use the trapezium rule with 5 ordinates to find an estimate of $\int_0^{\pi} x \sin x \, \mathrm{d}x$ to 3 decimal places. [4 marks]

Q3 a) Show that $\int_0^{10} \ln(x + 1) \, \mathrm{d}x$ is approximately equal to $\ln(11 \times 945^2)$, using the trapezium rule with 5 strips. [4 marks]

b) The trapezium rule with 10 strips is used to find another estimate to the integral in part a). Show that a lower bound for this estimate is $\ln(10!)$. [4 marks]

Numerical integration is the limit of my concentration span...

It can seem like a bit of a faff to work out all those y-values to stick in the trapezium rule formula — but it's often much simpler than integrating a messy function, and reasonably accurate too if you use enough strips. Unfortunately, you might have to do both in an exam question, and compare the answers. Don't say I didn't warn you...