Integrating Using the Chain Rule Backwards

Most integrations aren't as bad as they look — a few pages ago, you saw how to integrate special fractions, and now it's time for certain products. There are some things you can look out for when you're integrating...

*You can use the **Chain Rule** in **Reverse***

You came across the **chain rule** back on p.94 — it's where you write the thing you're differentiating in terms of u (and u is a **function** of x). You end up with the **product** of **two derivatives** ($\frac{dy}{du}$ and $\frac{du}{dx}$). When it comes to integrating, if you spot that your integral is a **product** where one bit is the **derivative** of another bit, you can use this rule:

$$\int \frac{du}{dx} f'(u)\,dx = f(u) + C$$

where u is a function of x.

Examples: Find: a) $\int 6x^5 e^{x^6}\,dx$, b) $\int e^{\sin x}\cos x\,dx$.

a) $\int 6x^5 e^{x^6}\,dx = \mathbf{e^{x^6} + C}$ If you differentiated $y = e^{x^6}$ using the chain rule, you'd get $6x^5e^{x^6}$, which is the function you had to integrate. In the formula, $u = x^6$ and $f(u) = e^u$.

b) $\int e^{\sin x}\cos x\,dx = \mathbf{e^{\sin x} + C}$ If you differentiated $y = e^{\sin x}$ using the chain rule, you'd get $e^{\sin x}\cos x$, which is the function you had to integrate. Again, $f(u) = e^u$ and $u = \sin x$.

*Some **Products** are made up of a **Function** and its **Derivative***

Similarly, if you spot that part of a **product** is the **derivative** of the other part of it (which is raised to a **power**), you can integrate it using this **rule**:

$$\int (n+1)f'(x)[f(x)]^n\,dx = [f(x)]^{n+1} + C$$

Remember that the **derivative** will be a **multiple** of $n + 1$ (not n) — watch out for any other multiples too. This will probably make more sense if you have a look at an **example**:

Examples: Find: a) $\int 12x^3(2x^4 - 5)^2\,dx$, b) $\int 8\,\text{cosec}^2 x \cot^3 x\,dx$.

This one looks pretty horrific, but it isn't too bad once you spot that $-\text{cosec}^2 x$ is the derivative of $\cot x$.

a) Here, $f(x) = 2x^4 - 5$, so differentiating gives $f'(x) = 8x^3$. $n = 2$, so $n + 1 = 3$.

Putting all this into the rule above gives: $\int 3(8x^3)(2x^4 - 5)^2\,dx = \int 24x^3(2x^4 - 5)^2\,dx = (2x^4 - 5)^3 + C$

Divide everything by 2 to match the original integral: $\int 12x^3(2x^4 - 5)^2\,dx = \mathbf{\frac{1}{2}(2x^4 - 5)^3 + C}$

b) For this one, $f(x) = \cot x$, so differentiating gives $f'(x) = -\text{cosec}^2 x$. $n = 3$, so $n + 1 = 4$.

Putting all this into the rule gives: $\int -4\,\text{cosec}^2 x \cot^3 x\,dx = \cot^4 x + C$

Multiply everything by -2 to match the original integral: $\int 8\,\text{cosec}^2 x \cot^3 x\,dx = \mathbf{-2\cot^4 x + C}$

Practice Questions

Q1 Integrate a) $\int 3x^2 e^{x^3}\,dx$ b) $\int 8x\cos(x^2)\,dx$.

Q2 Find $\int 36x^2(3x^3 + 4)^3\,dx$.

Exam Questions

Q1 Integrate the function $f(x) = -\sin x\,(3\cos^2 x)$ with respect to x. [3 marks]

Q2 Find the indefinite integral $\int 6x(\text{cosec}^2 x^2 - \sec x^2 \tan x^2)\,dx$. [6 marks]

To get rid of hiccups, drink a glass of water backwards...

It seems to me that most of this section is about reversing the things you learnt in Section 7. I don't know why they ask you to differentiate stuff if you're just going to have to integrate it right back again. Well, at least it keeps you busy...

Integration by Substitution

So, I know we said that the stuff on the last page was the chain rule backwards, but (and bear with me here) integration by substitution is also the reverse of the chain rule. Just a different reverse. You know what, just read on...

Use *Integration by Substitution* on *Products* of *Two Functions*

On the previous page, you saw how to **integrate** certain **products of functions** using the **chain rule backwards**. **Integration by substitution** lets you integrate **functions of functions** by **simplifying** the **integral**. Like the chain rule, you have to write part of the function in terms of u, where u is some **function** of x.

Integration by Substitution

1) You'll be given an integral that's made up of two functions of x (one is often just x) — e.g. $x(3x + 2)^3$.
2) Substitute u for one of the functions of x (to give a function that's easier to integrate) — e.g. $u = 3x + 2$.
3) Next, find $\frac{du}{dx}$, and rewrite it so that dx is on its own — e.g. $\frac{du}{dx} = 3$, so $dx = \frac{1}{3}\,du$.
4) Rewrite the original integral in terms of u and du — e.g. $\int x(3x + 2)^3\,dx$ becomes $\int \left(\frac{u-2}{3}\right)u^3\frac{1}{3}\,du = \int \frac{u^4 - 2u^3}{9}\,du$.
5) You should now be left with something that's easier to integrate — just integrate as normal, then at the last step replace u with the original substitution (so for this one, replace u with $3x + 2$).

You won't always be told what substitution to use, but bits inside brackets or square roots are a good place to start.

$\frac{du}{dx}$ isn't really a fraction, but you can treat it as one for this bit.

Example: Use the substitution $u = x^2 - 2$ to find $\int 4x(x^2 - 2)^4\,dx$.

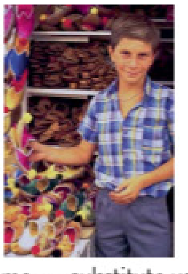
As $u = x^2 - 2$, $\frac{du}{dx} = 2x$, so $dx = \frac{1}{2x}du$.

Substituting gives $\int 4x(x^2 - 2)^4\,dx = \int 4xu^4\frac{1}{2x}du = \int 2u^4\,du$.

Integrate... $\int 2u^4\,du = \frac{2}{5}u^5 + C$

...and substitute x back in: $= \frac{2}{5}(x^2 - 2)^5 + C$

The x's cancel, making it a lot easier to integrate — this often happens.



"Come — substitute your boring old shoes for one of my beautiful products."

For *Definite Integrals*, you have to *Change* the *Limits*

If you're given a **definite integral**, it's really important that you remember to **change the limits** to u. Doing it this way means you **don't** have to **put x back in** at the last step — just put the numbers into the integration for u.

Example: Use the substitution $u = \cos x$ to find $\int_{\frac{\pi}{2}}^{2\pi} -12\sin x\cos^3 x\,dx$.

As $u = \cos x$, $\frac{du}{dx} = -\sin x$, so $dx = -\frac{1}{\sin x}du$.

Find the limits of u: When $x = \frac{\pi}{2}$, $u = \cos\frac{\pi}{2} = 0$, when $x = 2\pi$, $u = \cos 2\pi = 1$.
So the limits of u are 0 and 1.

Make sure you keep the top limit on the top and the bottom limit on the bottom — even if the bottom limit ends up as the bigger one.

Substituting all this gives: $\int_{\frac{\pi}{2}}^{2\pi} -12\sin x\cos^3 x\,dx = \int_0^1 -12\sin x\,u^3\frac{-1}{\sin x}du = \int_0^1 12u^3\,du$

Integrating and putting in the values of the limits gives: $\int_0^1 12u^3\,du = [3u^4]_0^1 = [3(1)^4] - [3(0)^4] = 3$

You could also have solved this one using the method on the previous page.

Integration by Substitution

You can use **Integration by Substitution** on **Fractions**

When faced with an **algebraic fraction** to integrate, you might be able to use the methods on pages 114-115. However, sometimes it's a lot easier to use a **substitution** (usually with u = the **denominator** or something in the denominator) to simplify the fraction into something you can integrate with ease.

Example: Find $\int \frac{3x^2}{(2x^3-1)^{\frac{1}{3}}}\,dx$, using a suitable substitution.

You're not given a substitution to use here, so pick something that looks like it might help. The $(2x^3 - 1)$ in the denominator is a sensible choice.

Let $u = (2x^3 - 1)$, then $\frac{du}{dx} = 6x^2$, so $dx = \frac{1}{6x^2}\,du$

Substituting gives: $\int \frac{3x^2}{(2x^3-1)^{\frac{1}{3}}}\,dx = \int \frac{3x^2}{u^{\frac{1}{3}}} \times \frac{1}{6x^2}\,du = \int \frac{1}{2}u^{-\frac{1}{3}}\,du$

Now you can integrate: $\int \frac{1}{2}u^{-\frac{1}{3}}\,du = \frac{1}{2}\left(\frac{3}{2}u^{\frac{2}{3}}\right) + C = \mathbf{\frac{3}{4}(2x^3-1)^{\frac{2}{3}} + C}$

You could also use the method from p.119 here, using $n = -\frac{1}{3}$

Some **Trig Integrals** can be really **Nasty**

Unfortunately, the vast range of **trig identities** and **formulas** you've seen can all show up in one of these questions, meaning that there's no end to the **evil integration questions**. Here's a particularly nasty example:

Example: Use the substitution $u = \tan x$ to find $\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$.

First, work out what all the substitutions will be: If $u = \tan x$, then $\frac{du}{dx} = \sec^2 x$, so $dx = \frac{1}{\sec^2 x}\,du$.

This will leave $\sec^2 x$ on the numerator — you need to find this in terms of u: From the identity $\sec^2 x \equiv 1 + \tan^2 x$, you get $\sec^2 x \equiv 1 + u^2$.

Then substitute all these bits into the integral:

$$\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx = \int \left(\frac{\sec^4 x}{\sqrt{\tan x}} \times \frac{1}{\sec^2 x}\right) du = \int \frac{1+u^2}{\sqrt{u}}\,du$$

$$= \int \left(\frac{1}{\sqrt{u}} + \frac{u^2}{\sqrt{u}}\right) du = \int \left(u^{-\frac{1}{2}} + u^{\frac{3}{2}}\right) du$$

Remember to stick u = tan x back into the equation.

$$= 2u^{\frac{1}{2}} + \frac{2}{5}u^{\frac{5}{2}} + C = \mathbf{2\sqrt{\tan x} + \frac{2}{5}\sqrt{\tan^5 x} + C}$$

Practice Questions

Q1 Use the substitution $u = e^x - 1$ to find $\int e^x(e^x+1)(e^x-1)^2\,dx$.

Q2 Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^4 x \tan x\,dx$, using the substitution $u = \sec x$.

Q3 Use a suitable substitution to find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos x + 2)^3}\,dx$.

Exam Questions

Q1 Find the value of $\int_1^2 \frac{8}{x}(\ln x + 2)^3\,dx$ using the substitution $u = \ln x$. Give your answer to 4 s.f. [6 marks]

Q2 a) Use a suitable substitution to show that $\int \frac{1}{\sqrt{u}(\sqrt{u}-1)^2}\,du = -\frac{2}{(\sqrt{u}-1)} + C$ [4 marks]

b) Hence use the substitution $u = x^3$ to find $\int \frac{3\sqrt{x}}{(x^{\frac{3}{2}}-1)^2}\,dx$ in terms of x. [4 marks]

Maths Pick-Up Line #7 — I wanna substitute my x for u...

Life is full of limits — time limits, height limits, limits of how many times I can gaze at my Hugh Jackman poster while still getting my work done... But at least limits of integration will get you exam marks, so it's worth practising them.