# Integration by Parts

*Just like you can differentiate products using the product rule (see p.100), you can integrate products using... er... integration by parts. Not quite as catchy I know, but just as thrilling.*

## ***Integration by Parts*** *is the* ***Reverse*** *of the* ***Product Rule***

If you have to integrate a **product** but can't use integration by substitution (see the previous pages), you might be able to use **integration by parts**. The **formula** for integrating by parts is:

This is on the formula sheet — and rightly so.

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

where $u$ and $v$ are both functions of $x$.

...or for definite integrals,

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

The hardest thing about integration by parts is **deciding** which bit of your product should be $u$ and which bit should be $\frac{dv}{dx}$. There's no set rule for this — you just have to look at both parts and see which one **differentiates** to give something **nice**, then set that one as $u$. For example, if you have a product that has a **single** $x$ as one part of it, choose this to be $u$. It differentiates to **1**, which makes **integrating** $v\frac{du}{dx}$ dead easy.

If you have a product that has ln x as one of its factors, let u = ln x, as ln x is easy to differentiate but quite tricky to integrate (see below).

**Examples:** Find: a) $\int 2xe^x\,dx$, b) $\int_{\frac{\pi}{2}}^{\pi} x\sin x\,dx$.

a) Let $u = 2x$ and let $\frac{dv}{dx} = e^x$.

Then $u$ differentiates to give $\frac{du}{dx} = 2$ and $\frac{dv}{dx}$ integrates to give $v = e^x$.

Put these into the integration by parts formula: $\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx$

$= \mathbf{2xe^x - 2e^x + C}$

b) Let $u = x$ and let $\frac{dv}{dx} = \sin x$.

Then $u$ differentiates to give $\frac{du}{dx} = 1$ and $\frac{dv}{dx}$ integrates to give $v = -\cos x$.

Putting these into the formula gives: $\int_{\frac{\pi}{2}}^{\pi} x\sin x\,dx = [-x\cos x]_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi}(-\cos x)\,dx$

$= [-x\cos x]_{\frac{\pi}{2}}^{\pi} + [\sin x]_{\frac{\pi}{2}}^{\pi}$

$= \left[(-\pi\cos\pi) - \left(-\frac{\pi}{2}\cos\frac{\pi}{2}\right)\right] + \left[\sin\pi - \sin\frac{\pi}{2}\right]$

$= [\pi - 0] + [0 - 1] = \mathbf{\pi - 1}$

## *You can integrate* ***ln x*** *using* ***Integration by Parts***

Up till now, you haven't been able to integrate **ln** $x$, but all that is about to change.
There's a little trick you can use — write $(\ln x)$ as $(1 \times \ln x)$ then **integrate by parts**.

**Examples:** Find: a) $\int \ln x\,dx$, b) $\int (9x^2 - 2)\ln x\,dx$.

a) To find $\int \ln x\,dx$, write $\ln x = 1 \times \ln x$.

Let $u = \ln x$ and let $\frac{dv}{dx} = 1$.

Then $u$ differentiates to give $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx}$ integrates to give $v = x$.

Putting these into the formula gives: $\int \ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx$

$= x\ln x - \int 1\,dx = \mathbf{x\ln x - x + C}$

b) Let $u = \ln x$ and let $\frac{dv}{dx} = 9x^2 - 2$.

Then $u$ differentiates to give $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx}$ integrates to give $v = 3x^3 - 2x$.

Putting these into the formula gives: $\int (9x^2 - 2)\ln x\,dx = (3x^3 - 2x)\ln x - \int (3x^3 - 2x)\frac{1}{x}\,dx$

$= (3x^3 - 2x)\ln x - \int (3x^2 - 2)\,dx$

$= (3x^3 - 2x)\ln x - (x^3 - 2x) + C$

$= \mathbf{(3x^3 - 2x)\ln x - x^3 + 2x + C}$

# Integration by Parts

## You might have to integrate by parts More Than Once

If you have an integral that **doesn't** produce a nice, easy-to-integrate function for $v\frac{du}{dx}$, you might have to carry out integration by parts **more than once**.

**Examples:** Find: a) $\int 3x^2e^x\,dx$, b) $\int x^2 \sin x$.

a) Let $u = 3x^2$ and let $\frac{dv}{dx} = e^x$.

Then $u$ differentiates to give $\frac{du}{dx} = 6x$ and $\frac{dv}{dx}$ integrates to give $v = e^x$.

Putting these into the formula gives: $\int 3x^2e^x\,dx = 3x^2e^x - \int 6xe^x\,dx$

To work out $\int 6xe^x\,dx$, use integration by parts again:

> Let $u = 6x$ and let $\frac{dv}{dx} = e^x$. Then $\frac{du}{dx} = 6$ and $v = e^x$.
>
> Putting these into the formula gives: $\int 6xe^x\,dx = 6xe^x - \int 6e^x\,dx = 6xe^x - 6e^x$

So $\int 3x^2e^x\,dx = 3x^2e^x - (6xe^x - 6e^x) + C = 3x^2e^x - 6xe^x + 6e^x + C$

b) Let $u = x^2$ and let $\frac{dv}{dx} = \sin x$.

Then $u$ differentiates to give $\frac{du}{dx} = 2x$ and $\frac{dv}{dx}$ integrates to give $v = -\cos x$.

Putting these into the formula gives: $\int x^2 \sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$

$$= -x^2\cos x + \int 2x\cos x\,dx$$

$2x\cos x$ isn't very easy to integrate, so integrate by parts again:

> Let $u = 2x$ and let $\frac{dv}{dx} = \cos x$. Then $\frac{du}{dx} = 2$ and $v = \sin x$.
>
> Putting these into the formula gives: $\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx$
>
> $= 2x\sin x + 2\cos x$

So $\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

## Practice Question

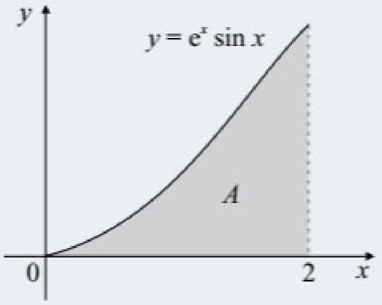
Q1 Use integration by parts to find: a) $\int 3x^2\ln x\,dx$, b) $\int 4x\cos 4x\,dx$, c) $\int 8xe^{2x}\,dx$.

**Exam Question**

Q1 A park occupies the area between a straight road modelled by the function $y = 0$, and another road which follows a path modelled by the function $y = e^x\sin x$, as shown in the diagram as area $A$.

a) Use integration by parts twice to show that the area of the park satisfies the equation:
$A = [e^x\sin x]_0^2 - [e^x\cos x]_0^2 - A$ [5 marks]

b) Hence find the area of the park between the boundaries at $x = 0$ and $x = 2$, correct to 3 s.f. [2 marks]




## Not those 'parts', sunshine — put 'em away...

*After you've had a go at some examples, you'll probably realise that integrals with $e^x$, sin x or cos x in them are actually quite easy, as all three are really easy to integrate and differentiate. Fingers crossed you get one of them in the exam.*