Differential Equations

Differential equations are tricky little devils that have a lot to do with differentiation as well as integration. They're often about rates of change, so the variable t pops up quite a lot.

Differential Equations have a dy/dx Term

(or $\frac{dP}{dt}$, $\frac{ds}{dt}$, $\frac{dV}{dr}$, etc. — depending on the variables)

1) A **differential equation** is one that has a **derivative term** (such as $\frac{dy}{dx}$), as well as **other terms** (like x and y).
2) Before you even think about **solving** them, you have to be able to **set up** ('**formulate**') differential equations.
3) Differential equations tend to involve a **rate of change** (giving a derivative term) and a **proportion relation**. Remember — if $a \propto b$, then $a = kb$ for some **constant** k (see p.33).

Example: The number of bacteria in a petri dish is increasing over time, t, at a rate directly proportional to the number of bacteria at the time, b. Formulate a differential equation to show this information.

The rate of change, $\frac{db}{dt}$, is proportional to b, so $\frac{db}{dt} \propto b$. This means that $\mathbf{\frac{db}{dt} = kb}$ for some constant k, $k > 0$.

Example: The volume of interdimensional space jelly, V, in a container is decreasing over time, t, at a rate directly proportional to the square of its volume. Show this as a differential equation.

The rate of change, $\frac{dV}{dt}$, is proportional to V^2, so $\frac{dV}{dt} \propto V^2$. $\mathbf{\frac{dV}{dt} = -kV^2}$ for some constant k, $k > 0$.

V is decreasing, so don't forget the –.

Solve differential equations by Integrating

Now comes the really juicy bit — **solving** differential equations. It's not as bad as it looks (honest).

Solving Differential Equations

1) You can only solve differential equations if they have separable variables — where x and y can be separated into functions $f(x)$ and $g(y)$.
2) Write the differential equation in the form $\frac{dy}{dx} = f(x)g(y)$.
3) Then rearrange the equation to get all the terms with y on the left-hand side and all the terms with x on the right-hand side. It'll look something like this: $\frac{1}{g(y)}\,dy = f(x)\,dx$.
4) Now integrate both sides: $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$.
5) Rearrange your answer to get it in a nice form — you might be asked to find it in the form $y = h(x)$. Don't forget the constant of integration (you only need one — not one on each side). It might be useful to write the constant as ln k rather than C (see p.115).
6) If you're asked for a general solution, leave C (or k) in your answer. If they want a particular solution, they'll give you x and y values for a certain point. All you do is put these values into your equation and use them to find C (or k).

Remember — it might not be in terms of x and y.

Like in integration by substitution, you can treat dy/dx as a fraction here.

Example: Find the particular solution of $\frac{dy}{dx} = 2y(1+x)^2$ when $x = -1$ and $y = 4$.

This equation has separable variables: $f(x) = 2(1+x)^2$ and $g(y) = y$.

Rearranging this equation gives: $\frac{1}{y}\,dy = 2(1+x)^2\,dx$

And integrating: $\int \frac{1}{y}\,dy = \int 2(1+x)^2\,dx \Rightarrow \ln|y| = \frac{2}{3}(1+x)^3 + C$

Now put in the values of x and y to find the value of C:

$$\ln|4| = \frac{2}{3}(1+(-1))^3 + C \Rightarrow \ln 4 = C$$

So the particular solution is $\mathbf{\ln|y| = \frac{2}{3}(1+x)^3 + \ln 4}$

If you were asked for a general solution, you could just leave it in this form.

You could be asked to sketch members of the family of solutions of a differential equation — this just means the graph of the general solution, for a few different values of C (or k). Graph transformations (see p.31) are very handy for figuring out what they look like.

Differential Equations

You might be given Extra Information

1) In the exam, you might be given a question that uses differential equations to **model** a **real-life problem**.
2) **Population** questions are an example of this — the population might be **increasing** or **decreasing**, and you have to find and solve differential equations to show it. In cases like this, one of the variables will usually be t, **time**.
3) You might be given a **starting condition** — e.g. the **initial population**. The important thing to remember is that:

   The starting condition occurs when $t = 0$. ← This is pretty obvious, but it's really important.

4) You might also be given **extra information** — e.g. a **specific population** (where you have to figure out what t is when the population reaches this number), or a **specific time** (where you have to work out what the population will be at this time). Make sure you always **link** the numbers you get back to the **situation**.

Example: The population of rabbits in a park is decreasing as winter approaches. The decrease in the population, P, after t days, is modelled by the differential equation $\frac{dP}{dt} = -0.1P$. Find the time at which the population of rabbits will have halved, to the nearest day.

First, solve the differential equation to find the general solution: $\frac{dP}{dt} = -0.1P \Rightarrow \frac{1}{P}\,dP = -0.1\,dt$

Integrating this gives: $\int \frac{1}{P}\,dP = \int -0.1\,dt \Rightarrow \ln P = -0.1t + C$ ← You don't need modulus signs for ln P as P ≥ 0 — you can't have a negative population.

At $t = 0$, $P = P_0$. Putting these values into the equation gives: $\ln P_0 = -0.1(0) + C \Rightarrow \ln P_0 = C$

So the differential equation becomes: $\ln P = -0.1t + \ln P_0 \Rightarrow P = e^{(-0.1t + \ln P_0)} = e^{-0.1t}e^{\ln P_0} \Rightarrow P = P_0e^{-0.1t}$

When the population of rabbits has halved, $P = \frac{1}{2}P_0$: (Remember that $e^{\ln x} = x = \ln e^x$.)

$$\frac{1}{2}P_0 = P_0e^{-0.1t} \Rightarrow \frac{1}{2} = e^{-0.1t} \Rightarrow \ln\frac{1}{2} = -0.1t \Rightarrow -0.6931... = -0.1t \Rightarrow t = 6.931...$$

So, to the nearest day, it will take **7 days** for the population of rabbits to halve.

You could also be asked to talk about **limitations** of a model, and suggest possible **changes** that would **improve** it. Think about things like:

- **missing information** (e.g. **above**, you weren't told P_0),
- what happens for really **big/small** values of the variables (e.g. as t gets large, P gets small but **never** reaches 0),
- how **appropriate** the model is (e.g. $P_0e^{-0.1t}$ is a **continuous** function, but population is a **discrete** variable),
- **any other factors** that haven't been included (e.g. what happens to the poor rabbits when **winter** arrives).

Practice Question

Q1 Find the general solution to the following differential equations, giving your answers in the form $y = f(x)$:

a) $\frac{dy}{dx} = \frac{1}{y}\cos x \quad (y > 0)$ b) $\frac{dy}{dx} = 2y^2 - 3(xy)^2$ c) $\frac{dy}{dx} = e^{x-y}$

Exam Questions

Q1 a) Find the general solution to the differential equation $\frac{dy}{dx} = \frac{\cos x \cos^2 y}{\sin x}$, $0 < x < \pi$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$. [4 marks]

b) Given that $y = 0$ when $x = \frac{\pi}{6}$, solve the differential equation above. [2 marks]

Q2 A company sets up an advertising campaign to increase sales of margarine. After the campaign, the number of tubs of margarine sold each week, m, increases over time, t (in weeks), at a rate that is directly proportional to the square root of the number of tubs sold.

a) Formulate a differential equation in terms of t, m and a constant k. [2 marks]

b) At the start of the campaign, the company was selling 900 tubs of margarine a week. Use this information to solve the differential equation, giving m in terms of k and t. [5 marks]

c) Hence calculate the number of tubs sold in the fifth week after the campaign, given that $k = 2$. [2 marks]

d) Explain why the model is not likely to be accurate for large values of t. [2 marks]

At t = 10, we kill all the bunnies...

These questions can get a bit morbid — just how I like them. They might look a bit scary, as they throw a lot of information at you in one go, but once you know how to solve them, they're a walk in the park. Rabbit traps optional.