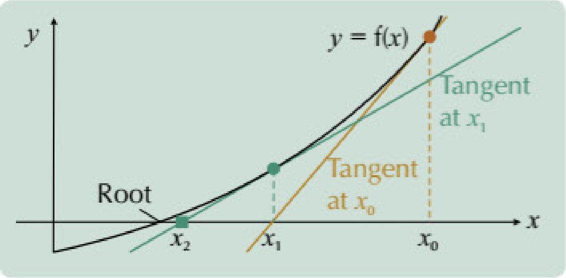
Iterative Methods

The ***Newton-Raphson Method*** *uses* ***Differentiation***

There's another method you can use to find a root — it's called the **Newton-Raphson method**.
To find **roots** of an equation in the form **f(x) = 0**, find **f′(x)** then use this formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

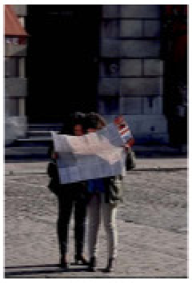
It works by finding the ***x*-intercept** of the **tangent** to the graph of f(x) at x_n, and using this value as x_{n+1}. As you can see from the diagram below, this will (usually) get you closer and closer to the root:



Beware: there are a few things that can cause the Newton-Raphson method to fail to find a root — see p.132 for more on this.

Example: Find a root of the equation $x^2 \ln x = 5$ to 5 s.f., using $x_0 = 2$.

1) First, rearrange the equation so it's in the form $f(x) = 0$: $x^2 \ln x - 5 = 0$.
2) Next, find $f'(x)$:
   Using the product rule (see p.100), let $u = x^2$, so $\frac{du}{dx} = 2x$. Let $v = \ln x$, so $\frac{dv}{dx} = \frac{1}{x}$.
   Putting this into the product rule formula gives: $\frac{dy}{dx} = x(1 + 2\ln x)$.
3) The formula for the Newton-Raphson method is $x_{n+1} = x_n - \frac{x_n^2 \ln x_n - 5}{x_n(1 + 2\ln x_n)}$.
4) Starting with $x_0 = 2$, this gives $x_1 = 2 - \frac{2^2 \ln 2 - 5}{2(1 + 2\ln 2)} = 2.466709...$
5) Further iterations give:
   $x_2 = 2.395369...$, $x_3 = 2.393518...$, $x_4 = 2.393517...$, $x_5 = 2.393517...$
6) So a root of $x^2 \ln x = 5$ is 2.3935 (5 s.f.).



Unfortunately, Newton and Raphson were less proficient at finding routes.

Practice Questions

Q1 Use the formula $x_{n+1} = \sqrt{\ln x_n + 4}$, with $x_0 = 2$, to find a root of $x^2 - \ln x - 4 = 0$ to 3 d.p.

Q2 a) Show that the equation $2x^2 - x^3 + 1 = 0$ can be written in the form:
(i) $x = \sqrt{\frac{-1}{2-x}}$, (ii) $x = \sqrt[3]{2x^2 + 1}$, (iii) $x = \sqrt{\frac{x^3 - 1}{2}}$.

b) Use iteration formulas based on each of the above rearrangements with $x_0 = 2.3$ to find a root of $2x^2 - x^3 + 1 = 0$ to 2 d.p. Which of the three formulas converge to a root?

Q3 Use the Newton-Raphson method to find a root of $x^4 - 2x^3 = 5$ to 5 s.f., starting with $x_0 = 2.5$.

Exam Question

Q1 a) Show that the equation $\sin 3x + 3x = 1$ can be written as $x = \frac{1}{3}(1 - \sin 3x)$. [1 mark]

b) Starting with $x_0 = 0.2$, use the iteration $x_{n+1} = \frac{1}{3}(1 - \sin 3x_n)$ to find x_4 in radians to 3 d.p. [2 marks]

c) Use the Newton-Raphson method to find the same root to 3 d.p., starting with $x_0 = 0.2$ again. Comment on which method was more effective. [5 marks]

The hat — an approximate solution to root problems...

Just to re-iterate (ho ho) — you can often find a decent approximation to a root by using an iteration formula. Either rearrange the equation to get a single x on one side, or differentiate it and use the Newton-Raphson formula.

More on Iterative Methods

Exam Questions combine all the Different Methods

In exam questions, you might have to use **different methods** to find roots — like in this giant worked example.

Example: The graph below shows both roots of the continuous function $f(x) = 6x - x^2 + 13$.

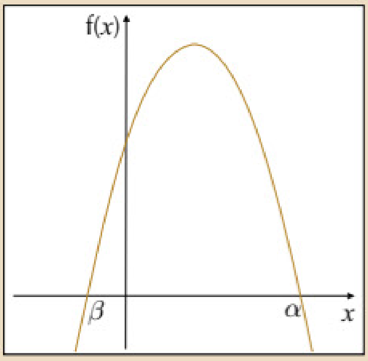
a) Show that the positive root, α, of $f(x) = 0$ lies in the interval $7 < x < 8$.

b) Show that $6x - x^2 + 13 = 0$ can be rearranged into the formula $x = \sqrt{6x + 13}$.

c) Use the iteration formula $x_{n+1} = \sqrt{6x_n + 13}$ and $x_0 = 7$ to find α to 1 d.p.

d) Sketch a diagram to show the convergence of the sequence for x_1, x_2 and x_3.

e) Use the Newton-Raphson method to find the negative root, β, to 5 s.f. Start with $x_0 = -1$. Use bounds to check your root.



a) f(x) is a **continuous function**, so if f(7) and f(8) have **different signs** then there is a root in the interval $7 < x < 8$:

$f(7) = (6 \times 7) - 7^2 + 13 = 6.$

$f(8) = (6 \times 8) - 8^2 + 13 = -3.$

There is a **change of sign** so $7 < \alpha < 8$.

b) Get the x^2 on its own to make: $6x + 13 = x^2$

Now take the (positive) square root to leave: $x = \sqrt{6x + 13}$

The list of results from each iteration $x_1, x_2, x_3 ...$ is called the iteration sequence.

c) Using $x_{n+1} = \sqrt{6x_n + 13}$ with $x_0 = 7$, gives $x_1 = \sqrt{6 \times 7 + 13} = 7.4161...$

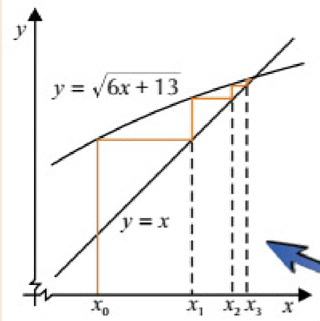
Continuing the iterations:

$x_2 = \sqrt{6 \times 7.4161... + 13} = 7.5826...$ $\quad x_3 = \sqrt{6 \times 7.5826... + 13} = 7.6482...$

$x_4 = \sqrt{6 \times 7.6482... + 13} = 7.6739...$ $\quad x_5 = \sqrt{6 \times 7.6739... + 13} = 7.6839...$

$x_6 = \sqrt{6 \times 7.6839... + 13} = 7.6879...$ $\quad x_7 = \sqrt{6 \times 7.6879... + 13} = 7.6894...$

x_4 to x_7 all round to **7.7 to 1 d.p.**, so to 1 d.p. $\alpha = 7.7$



d) Sketch $y = \sqrt{6x + 13}$ and $y = x$ on the same axes, and mark on the position of x_0. All you have to do is draw on the **lines** and label the **values** of x_1, x_2 and x_3. You can see from the diagram that the sequence is a **convergent staircase**.

e) Find $f'(x)$: $f'(x) = 6 - 2x$.

Putting this into the **Newton-Raphson formula** gives: $x_{n+1} = x_n - \dfrac{6x_n - x_n^2 + 13}{6 - 2x_n}$

Starting with $x_0 = -1$, this gives $x_1 = -1 - \dfrac{6(-1) - (-1)^2 + 13}{6 - 2(-1)} = -1.75$

Further iterations give $x_2 = -1.690789...$

$x_3 = -1.690415...$

$x_4 = -1.690415...$ So **$\beta = -1.6904$ to 5 s.f.**

The **upper and lower bounds** for this are −1.69035 and −1.69045, and $f(-1.69035) = 0.0006$, $f(-1.69045) = -0.0003$.

There is a **sign change** so the root is accurate to 5 s.f.

You might have to **compare** the different iteration methods — to do this, all you have to do is use each method to find the same root, then think about which one was the **easiest to use**, which was the **quickest** (i.e. took the fewest iterations), etc.

More on Iterative Methods

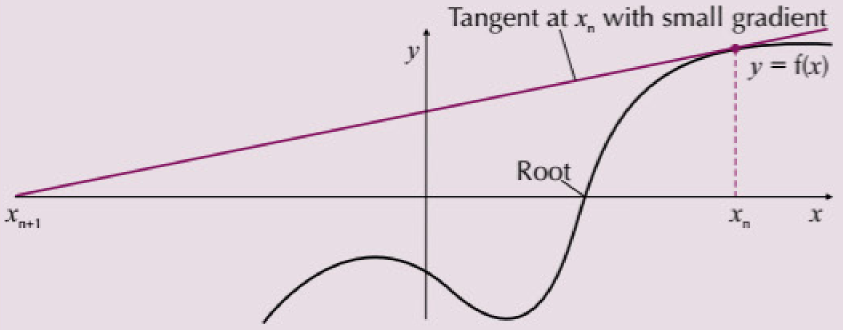
*The **Newton-Raphson** formula can **Stop Working***

Like with any iteration formula, if x_0 is **too far away** from the root, the **Newton-Raphson** method might give you a **divergent** iteration sequence.

But the Newton-Raphson formula also runs into problems when the **gradient of the tangent** at x_n is close to, or equal to, **zero**.

Another drawback of the Newton-Raphson method is that you can only use it if you can differentiate the function — which won't always be easy, or even possible.

$f'(x)$ close to 0



If you hit a point in the iteration sequence where the **tangent** to $f(x)$ has a very **shallow gradient**, the tangent meets the x-axis a **really long way away** from the root, which could make the sequence **diverge**.

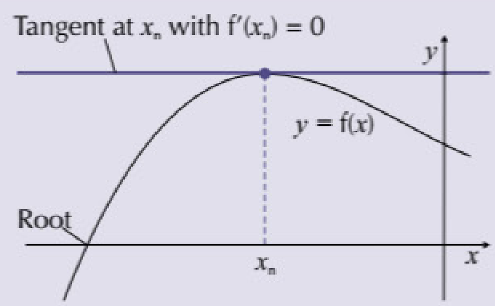
$f'(x) = 0$

If the tangent to $f(x)$ is **horizontal** at the point x_n (so x_n is a **stationary point**) then the method will **fail**. The tangent **never meets** the x-axis, so there's no x_{n+1} value for the next iteration.

The tangent is horizontal when **$f'(x) = 0$**, so the iteration formula would **stop working** at this point because you'd be **dividing by zero**:

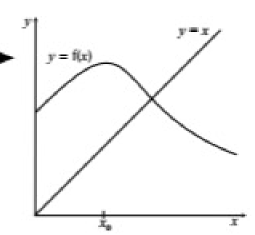
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

You won't get an answer if you divide by 0 here.



Practice Questions

Q1 Using the position of x_0 as given on the graph, draw a staircase or cobweb diagram showing how the sequence converges. Label x_1 and x_2 on the diagram.



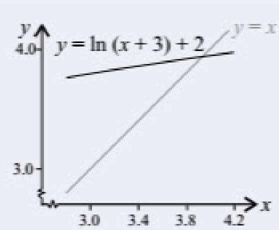
Q2 a) Show that the equation $x^4 - x^5 + 3 = 0$ has a root in the interval (1, 2), then find this root to 1 d.p. from a starting value of $x_0 = 1$, using:
   (i) the iteration formula $x_{n+1} = \sqrt[5]{x_n^4 + 3}$,
   (ii) the Newton-Raphson method.

b) Use bounds to check that your answer is a root to 1 d.p.

c) Compare the effectiveness of the two methods.

Exam Question

Q1 A graph plotted from experimental data is modelled as a function $f(x)$, where $f(x) = \ln(x + 3) - x + 2$, $x > -3$. $f(x)$ has a root at $x = m$.



a) Show that m lies between 3 and 4. [2 marks]

b) Find, using iteration, the value of m correct to 2 decimal places. Use the iteration formula $x_{n+1} = \ln(x_n + 3) + 2$, with $x_0 = 3$. [2 marks]

c) Show the convergence of the first 2 iterations found in b) on the diagram above. [2 marks]

d) Apply the Newton-Raphson method with $x_0 = 3$ to find m correct to 5 decimal places. [4 marks]

e) Explain why the Newton-Raphson method fails with $x_0 = -2$. [1 mark]

I feel like stopping working sometimes...

Just like the other iteration methods, the Newton-Raphson method might fail if you start too far away from the root. In the exam, you might have to show off your mad iteration skillz using all the different methods — so be prepared.