3D Vectors

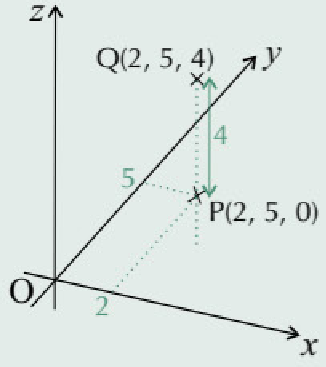
Vectors in 3D work the same as vectors in 2D, but the calculations and the diagrams can be a little trickier. Now, if you put on your (uncomfortable and overpriced) special glasses, this page is available in 3D...

*In **Three Dimensions** you use **Unit Vectors i, j and k***

1) Imagine that the x- and y-axes lie **flat** on the page. Then imagine a **third axis** sticking **straight through** the page at right angles to it — this is the **z-axis**.
2) The points in three dimensions are given (x, y, z) **coordinates**.
3) When you're talking vectors, **k** is the **unit vector** in the direction of the **z-axis**.
4) You can write three-dimensional vectors as **column vectors** like this:

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Column vectors work exactly the same way in three dimensions as they do in two dimensions.



Example: The point Q has coordinates (2, 5, 4). Write down $\overrightarrow{OQ}$ as a column vector.

The position vector of Q is given by its coordinates. It's two in the x-direction, 5 in the y-direction and 4 in the z-direction.

So, $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$.

Drawing diagrams for 3D vectors can be a little tricky — make sure you take your time and label everything carefully.

5) **3D vectors** work just like **2D vectors**, so all the things you saw on p.136 — vector addition and subtraction, multiplying by scalars, and showing if two vectors are parallel — apply to 3D vectors as well.

*You Can Use **Pythagoras** in **Three Dimensions** Too*

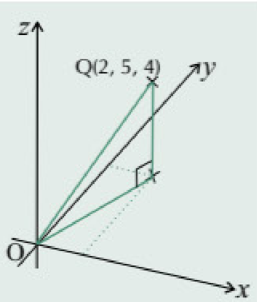
1) You can use a variation of **Pythagoras' theorem** to find the distance of any point in 3 dimensions from the origin, O.

The distance of point (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$.

2) This means that you can use Pythagoras' theorem to find the magnitude of a 3D vector the same way you used it in two dimensions.

Example (continued): Find $|\overrightarrow{OQ}|$.

$$|\overrightarrow{OQ}| = \sqrt{x^2 + y^2 + z^2}$$
$$= \sqrt{2^2 + 5^2 + 4^2}$$
$$= \sqrt{45}$$
$$= \textbf{6.7 units} \text{ (1 d.p.)}$$



Example: Find the magnitude of the vector $\mathbf{r} = 5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$ to 1 d.p.

$|\mathbf{r}| = \sqrt{5^2 + 7^2 + 3^2} = \sqrt{83} = \textbf{9.1 units}$ (1 d.p.)

3) There's also a Pythagoras-based formula for finding **the distance between any two points**.

The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Example: The position vector of point A is $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, and the position vector of point B is $2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$. Find $|\overrightarrow{AB}|$.

A has the coordinates (3, 2, 4), B has the coordinates (2, 6, –5).

$|\overrightarrow{AB}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \sqrt{(3-2)^2 + (2-6)^2 + (4-(-5))^2} = \sqrt{1 + 16 + 81} = \textbf{9.9 units}$ (1 d.p.)

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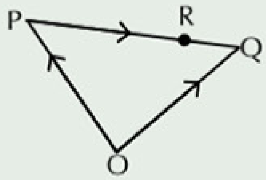
*Break up **3D Vector** Problems into **Smaller Chunks***

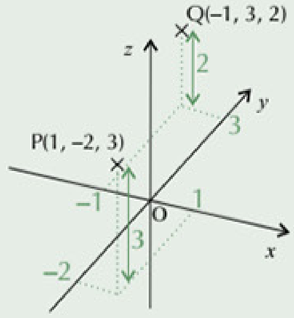
Visualising 3D vector problems can be quite hard. So, if you're given a **difficult** 3D vector problem with **multiple steps**, it helps to break it down into **chunks** and draw them with simple **2D diagrams**.

Example: Points P and Q have position vectors $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ respectively. Point R divides the line PQ in the ratio 3:1. Find the position vector of R.

First you need to find the vector $\overrightarrow{PQ}$:



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (-1 - 1)\mathbf{i} + (3 - (-2))\mathbf{j} + (2 - 3)\mathbf{k}$$
$$= -2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$$



R divides $\overrightarrow{PQ}$ in the ratio 3:1, so R is $\frac{3}{4}$ of the way along $\overrightarrow{PQ}$.

This means $\overrightarrow{PR} = \frac{3}{4}\overrightarrow{PQ} = \frac{3}{4}(-2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) = -\frac{3}{2}\mathbf{i} + \frac{15}{4}\mathbf{j} - \frac{3}{4}\mathbf{k}$.

So $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + -\frac{3}{2}\mathbf{i} + \frac{15}{4}\mathbf{j} - \frac{3}{4}\mathbf{k}$

$$= \left(1 - \frac{3}{2}\right)\mathbf{i} + \left((-2) + \frac{15}{4}\right)\mathbf{j} + \left(3 - \frac{3}{4}\right)\mathbf{k} = -\frac{1}{2}\mathbf{i} + \frac{7}{4}\mathbf{j} + \frac{9}{4}\mathbf{k}$$

You could also do $\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR}$.

Practice Questions

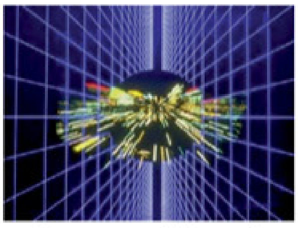
Q1 Point P has the coordinates (7, –3, –2). Find the position vector of P. Give your answer in **unit vector** form.

Q2 Find the **magnitudes** of these vectors: a) $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ b) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Q3 Show that the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$ are parallel.

Q4 Find $\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$ where $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Q5 The position vectors of point S and T are $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ respectively. Calculate the length of the line ST.



If vectors could dream, it would look something like this (probably)...

Exam Questions

Q1 Points P, Q and R have position vectors $\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix}$ respectively. Show that $\overrightarrow{OP}$ is parallel to $\overrightarrow{QR}$. [2 marks]

Q2 A skateboard ramp has vertices modelled with position vectors $O = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$ m, $X = (3\mathbf{i} - \mathbf{j})$ m, $Y = 5\mathbf{j}$ m and $Z = 4\mathbf{k}$ m. Calculate the distance from the midpoint of OZ to the midpoint of XY to 3 s.f. [3 marks]

Q3 a) The points A and B have position vectors $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $-3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ respectively. Show that $\overrightarrow{AB} = -5\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$. [1 mark]

b) The point M divides the line AB in the ratio 2:1. Calculate the distance of M from the origin. [5 marks]

Vectors let you flit between dimensions like your favourite sci-fi hero...

What do you mean you don't have a favourite sci-fi hero? Urgh, you haven't lived. Three dimensions doesn't really make things much more difficult — it just gives you an extra number to calculate with. You add, subtract and multiply 3D column vectors in the same way as 2D ones — you just have three rows to deal with.