$P(B \mid A)$ means Probability of B, given that A has Already Happened

Conditional probability means the probability of something, **given that** something else has **already happened**. For example, **$P(B \mid A)$** means the probability of B, given that A has already happened. Back to tree diagrams...

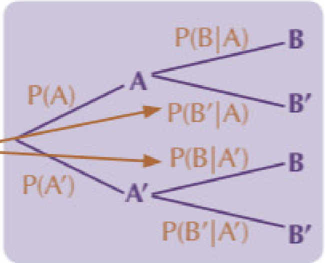
If you multiply probabilities along the branches, you get:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B \mid A)$$

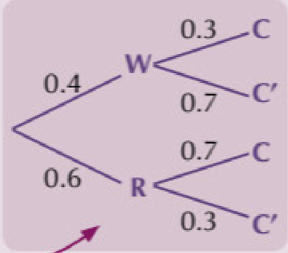
You can rewrite this as:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

These are conditional probabilities, since something (A or A′) has already happened.



Example: Horace either walks (W) or runs (R) to the bus stop. The probability that he walks to the bus stop is 0.4. If he walks, the probability that he catches the bus (C) is 0.3. If he runs, the probability that he catches the bus is 0.7. Find the probability that Horace catches the bus.



$$\begin{aligned} P(C) &= P(C \cap W) + P(C \cap R) \\ &= P(W)\,P(C \mid W) + P(R)\,P(C \mid R) \\ &= (0.4 \times 0.3) + (0.6 \times 0.7) = 0.12 + 0.42 = \mathbf{0.54} \end{aligned}$$

This is easier to follow if you match each part of this working to the probabilities in the tree diagram.

Tree Diagrams and Conditional Probability

*If **B is Conditional** on A then **A is Conditional** on B*

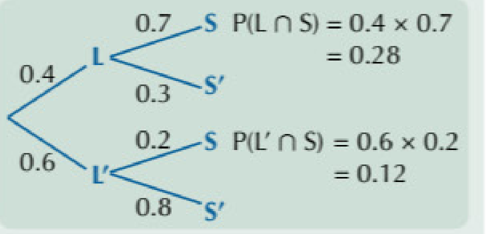
If B depends on A then A depends on B — regardless of which event happens first.

Example: Horace turns up at school either late (L) or on time (L′). He is then either shouted at (S) or not (S′). The probability that he turns up late is 0.4. If he turns up late the probability he is shouted at is 0.7. If he turns up on time the probability that he is shouted at is 0.2. If you hear Horace being shouted at, what is the probability that he turned up late?

1) The probability you want is $P(L \mid S)$. ← Get this the right way round — he's **already** being shouted at.

2) Use the conditional probability formula: $P(L \mid S) = \dfrac{P(L \cap S)}{P(S)}$

3) The best way to find $P(L \cap S)$ and $P(S)$ is with a tree diagram.



$$P(L \cap S) = 0.4 \times 0.7 = 0.28$$
$$P(S) = P(L \cap S) + P(L' \cap S) = 0.28 + 0.12 = 0.40$$

Be careful with questions like this — the information in the question tells you what you need to know to draw the tree diagram with L (or L′) considered first. But you need $P(L \mid S)$ — where S is considered first. So don't just rush in.

4) Put these in your conditional probability formula to get: $P(L \mid S) = \dfrac{0.28}{0.4} = 0.7$

Practice Questions

Q1 In a school orchestra (made up of pupils in either the upper or lower school), 40% of the musicians are boys. Of the boys, 30% are in the upper school. Of the girls in the orchestra, 50% are in the upper school.
- a) Represent this information on a tree diagram.
- b) Find the probability that a musician chosen at random is in the upper school.

Q2 For lunch, I eat either chicken or beef for my main course, and either chocolate cake or ice cream for dessert. The probability that I eat chicken is $\frac{1}{3}$, the probability that I eat ice cream given that I have chicken is $\frac{2}{5}$, and the probability that I have ice cream given that I have beef is $\frac{3}{4}$. Find the probability that:
- a) I have either chicken or ice cream — but not both, b) I eat ice cream,
- c) I had chicken, given that you see me eating ice cream.

Exam Questions

Q1 For a particular biased dice, the event 'throw a 6' is called event B. $P(B) = 0.2$. This biased dice and a fair dice are rolled together. Find the probability that:
- a) the biased dice doesn't show a 6, [1 mark]
- b) at least one of the dice shows a 6, [2 marks]
- c) exactly one of the dice shows a 6, given that at least one of them shows a 6. [3 marks]

Q2 A jar contains 3 red counters and 6 green counters. Three random counters are removed from the jar one at a time. The counters are not replaced after they are drawn.
- a) Draw a tree diagram to show the probabilities of the various outcomes. [3 marks]
- b) Find the probability that the third counter is green. [2 marks]
- c) Find the probability that all of the counters are the same colour. [2 marks]
- d) Find the probability that at least one counter is red. [2 marks]

Are you oak-ay, or pine-ing fir tree diagrams? (I willow-nly ash-k this once...)

Conditional probability questions can be a bit brain-twisting, but there's a fair chance that any probability question in the exam will have a conditional part to it — so it's worth doing a bit of practice to help get your head around them.

Mutually Exclusive and Independent Events

There's so much on these pages, it wouldn't all fit in the title — over on the right there's a bonus bit about modelling.

Mutually Exclusive Events have No Overlap

If two events **can't both happen** at the same time (i.e. $P(A \cap B) = 0$) they're called **mutually exclusive** (or just '**exclusive**'). If A and B are exclusive, then the probability of A **or** B is: $P(A \cup B) = P(A) + P(B)$.

This is the formula from p.152, with $P(A \cap B) = 0$.

More generally,

> For n **exclusive** events (i.e. only one of them can happen at a time):
> $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Example: Find the probability that a card pulled at random from a standard pack of cards (no jokers) is either a picture card (a Jack, Queen or King) or the 7, 8 or 9 of clubs.

Call **event A** — 'I get a picture card', and **event B** — 'I get the 7, 8 or 9 of clubs'. Then $P(A) = \frac{12}{52}$ and $P(B) = \frac{3}{52}$.
Events A and B are **mutually exclusive** — they can't both happen.
So the probability of either A or B is: $P(A \cup B) = P(A) + P(B) = \frac{12}{52} + \frac{3}{52} = \mathbf{\frac{15}{52}}$

Independent Events have No Effect on each other

If the probability of B happening doesn't depend on whether or not A has happened, then A and B are **independent**.

1) If A and B are independent, $\mathbf{P(A \mid B) = P(A)}$.
2) If you put this in the conditional probability formula, you get: $\mathbf{P(A \mid B) = P(A) = \frac{P(A \cap B)}{P(B)}}$

Or, to put that another way:

> For independent events: $\mathbf{P(A \cap B) = P(A)P(B)}$

Example: V and W are independent events, where $P(V) = 0.2$ and $P(W) = 0.6$. Find: a) $P(V \cap W)$, b) $P(V \cup W)$.

a) Just put the numbers into the formula for independent events: $P(V \cap W) = P(V)P(W) = 0.2 \times 0.6 = \mathbf{0.12}$
b) Using the formula on page 152: $P(V \cup W) = P(V) + P(W) - P(V \cap W) = 0.2 + 0.6 - 0.12 = \mathbf{0.68}$

Sometimes you'll be asked if two events are independent or not. Here's how you work it out...

Example: You are exposed to two infectious diseases — one after the other. The probability you catch the first (A) is 0.25, the probability you catch the second (B) is 0.5, and the probability you catch both of them is 0.2. Are catching the two diseases independent events?

Compare $P(A \mid B)$ and $P(A)$ — if they're different, the events **aren't independent**.

$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = 0.4$ $\qquad P(A) = 0.25$ $\qquad P(A \mid B)$ and $P(A)$ are different, so they're **not independent**.

Take Your Time with Tough Probability Questions

Example: A and B are two events, with $P(A) = 0.4$, $P(B \mid A) = 0.25$, and $P(A' \cap B) = 0.2$.
a) Find: (i) $P(A \cap B)$, (ii) $P(A')$, (iii) $P(B' \mid A)$, (iv) $P(B \mid A')$, (v) $P(B)$, (vi) $P(A \mid B)$.
b) Say whether or not A and B are independent.

a) (i) $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = 0.25$, so $P(A \cap B) = 0.25 \times P(A) = 0.25 \times 0.4 = \mathbf{0.1}$

(ii) $P(A') = 1 - P(A) = 1 - 0.4 = \mathbf{0.6}$

(iii) $P(B' \mid A) = 1 - P(B \mid A) = 1 - 0.25 = \mathbf{0.75}$

(iv) $P(B \mid A') = \frac{P(B \cap A')}{P(A')} = \frac{0.2}{0.6} = \mathbf{\frac{1}{3}}$ $\qquad P(B \cap A') = P(A' \cap B)$

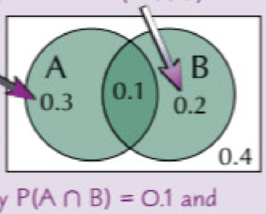
(v) $P(B) = P(B \mid A)P(A) + P(B \mid A')P(A') = (0.25 \times 0.4) + \left(\frac{1}{3} \times 0.6\right) = \mathbf{0.3}$ — Or use the Venn diagram.

(vi) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \mathbf{\frac{1}{3}}$

A Venn diagram can make it easier to see what's going on. Fill in the numbers as you work them out.

$P(A) = 0.4$ and from (i) $P(A \cap B) = 0.1$, so this must be 0.3.

$P(A' \cap B)$



b) If $P(B \mid A) = P(B)$, then A and B are independent.
But $P(B \mid A) = 0.25$, while $P(B) = 0.3$, so A and B are **not** independent.

Or you could say $P(A \cap B) = 0.1$ and $P(A)P(B) = 0.4 \times 0.3 = 0.12$ — they're different, so A and B are not independent.