Hypothesis Tests and Normal Distributions

If you like normal distributions and you like hypothesis testing, you're going to love this page.
If you need a reminder of the normal distribution, have a look back at Section 13.

Use a *Hypothesis Test* to *Find Out* about the *Population Mean*, μ

You can carry out hypothesis testing on the **mean** of a **normal distribution** too. Suppose $X \sim N(\mu, \sigma^2)$.
If you take a random sample of n observations from the distribution of X, and calculate the **sample mean** $\overline{X}$, you can use your **observed value** $\overline{x}$ to test theories about the **population mean** μ using the following method:

1) The **population parameter** you're testing will always be μ, the mean of the population.
2) The **null** hypothesis will be: $H_0: \mu = a$ for some constant a.
3) The **alternative** hypothesis, H_1, will either be $H_1: \mu < a$ or $H_1: \mu > a$ (one-tailed test)
   or $H_1: \mu \neq a$ (two-tailed test)
4) State the **significance level**, α — you'll usually be given this.
   Because the common significance levels are 10%, 5% and 1%, you can often use the table of percentage points for the normal distribution (see p.257).
5) To find the value of the **test statistic**:
   — Calculate the **sample mean**, $\overline{x}$.
   — If $X \sim N(\mu, \sigma^2)$, then $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$.
   — Then the value of your **test statistic** will be $z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$.
6) Use a **calculator** to test for significance, either by:
   — finding the **probability** of your test statistic taking a value **at least as extreme** as your observed value (the **p-value**) and comparing it to the significance level α.
   — finding the **critical value(s)** of the test statistic and seeing if your observed value lies in the **critical region**.
7) Write your **conclusion** — you'll either **reject H_0** or have **insufficient evidence** to do so.

Example: The times, in minutes, taken by the athletes in a running club to complete a certain run have been found to follow a N(12, 4) distribution. The coach increases the number of training sessions per week, and a random sample of 20 times run since the increase gives a mean time of 11.2 minutes. Assuming that the variance has remained unchanged, test at the 5% significance level whether there is evidence that the mean time has decreased.

Let μ = mean time since increase in training sessions. Then $H_0: \mu = 12$, $H_1: \mu < 12$, $\alpha = 0.05$.

*You assume that there's been **no change** in the value of the parameter (μ), so you can give it a value of 12.*

This is what you're looking to find evidence for.

Under H_0, $\overline{X} \sim N\left(12, \frac{4}{20}\right) \Rightarrow \overline{X} \sim N(12, 0.2)$ and $Z = \frac{\overline{X} - 12}{\sqrt{0.2}} \sim N(0, 1)$.

$$\overline{x} = 11.2 \Rightarrow z = \frac{11.2 - 12}{\sqrt{0.2}} = -1.789 \text{ (3 d.p.)}$$

This is a **one-tailed test** and you're interested in the lower end of the distribution. So the **critical value** is z such that $P(Z < z) = 0.05$. Using the **percentage points table**, you find that $P(Z < 1.645) = 0.95$ and so, by **symmetry**, $P(Z < -1.645) = 0.05$. So the critical value is -1.645 and the **critical region is $Z < -1.645$**.

If you want, you can instead do the test by working out the p-value, P(value at least as extreme as observed sample mean), and comparing it to α. So here you'd do:
$P(\overline{X} \leq 11.2) = P\left(Z \leq \frac{11.2 - 12}{\sqrt{0.2}}\right) = P(Z \leq -1.7888...)$
$= 0.03681... < 0.05$, so reject H_0.

Since $z = -1.789 < -1.645$, the **result is significant** and there is **evidence** at the 5% level of significance to **reject** H_0 and to suggest that the **mean time has decreased**.

Hypothesis Tests and Normal Distributions

For a *Two-Tailed Test*, divide α by 2

Example: The volume (in ml) of a cleaning fluid dispensed in each operation by a machine is normally distributed with mean μ and standard deviation 3.
Out of a random sample of 20 measured volumes, the mean volume dispensed was 30.9 ml.
Does this data provide evidence at the 5% level of significance that the machine is dispensing a mean volume that is different from 30 ml?

Let μ = mean volume (in ml) dispensed in all possible operations of the machine (i.e. μ is the mean volume of the 'population').

Your hypotheses will be: $\mathbf{H_0: \mu = 30}$ and $\mathbf{H_1: \mu \neq 30}$

The **significance level** is 5%, so $\alpha = \mathbf{0.05}$.

Under H_0, $X \sim N(30, 3^2)$, so $\overline{X} \sim N\left(30, \frac{3^2}{20}\right)$.

So under H_0, $Z = \frac{\overline{X} - 30}{3/\sqrt{20}} \sim N(0, 1)$.

Now find the value of your test statistic:

$\overline{x} = 30.9$, so $z = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} = \frac{30.9 - 30}{3/\sqrt{20}} = \mathbf{1.3416...}$

This is a **two-tailed** test, so you need to check whether the p-value is less than $\frac{\alpha}{2} = 0.025$.

$P(Z \geq 1.3416...) = \mathbf{0.0898... > \frac{\alpha}{2}}$

So the result is **not significant** at this level.

This data does **not** provide sufficient evidence at the 5% level to support the claim that the machine is dispensing a mean volume different from 30 ml.

You could work out the critical region instead — it's a two-tailed test, so the CR is given by $P(Z > z) = \frac{\alpha}{2} = 0.025$ or $P(Z < -z) = \frac{\alpha}{2} = 0.025$.
Using the percentage points table, this gives a value for z of 1.960.
So the critical region is $Z > 1.960$ or $Z < -1.960$.
Since $z = 1.3416... < 1.960$ (and > -1.960), there is insufficient evidence at the 5% level to reject H_0.

Practice Questions

Q1 Carry out the following test of the mean, μ, of a normal distribution with variance $\sigma^2 = 9$.
A random sample of 16 observations from the distribution was taken and the sample mean ($\overline{x}$) calculated.
Test H_0: $\mu = 45$ against H_1: $\mu < 45$, at the 5% significance level, using $\overline{x} = 42$.

Q2 A random sample of 10 observations is taken from a normal distribution with unknown mean μ and variance $\sigma^2 = 0.81$. The results are shown below.

20.1, 18.5, 19.6, 21.1, 20.7, 20.2, 19.5, 19.7, 20.2, 18.2

a) Calculate the value of the sample mean.

b) Carry out a hypothesis test, at the 5% level of significance, of the hypotheses H_0: $\mu = 20$, H_1: $\mu < 20$.

Exam Question

Q1 The heights of trees in an area of woodland are known to be normally distributed with a mean of 5.1 m and a variance of 0.2. A random sample of 100 trees from a second area of woodland is selected and the heights, X, of the trees are measured giving the following result:

$\sum x = 490$

a) Calculate the sample mean, $\overline{x}$, for the trees in this second area. [1 mark]

b) Test at the 0.1% level of significance whether the trees in the second area of woodland have a different mean height from the trees in the first area. [6 marks]

A statistician's party game — pin two tails on the donkey...

You can usually use either method to answer these questions (unless you're told which one to use — i.e. sometimes you might be asked to find the critical region). I personally prefer finding the critical region, as you can use the percentage points table which saves a bit of calculation — but it's entirely up to you. And that's this section done.