# Describing 2D Motion Using Vectors

## *Differentiate and Integrate the vector components Separately*

**Example:** A particle is moving on a horizontal plane so that at time $t$ it has velocity $\mathbf{v}$ ms$^{-1}$, where

$$\mathbf{v} = (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}$$

At $t = 2$, the particle has a position vector of $(10\mathbf{i} + 3\mathbf{j})$ m with respect to a fixed origin $O$.

a) Find the acceleration of the particle at time $t$.

b) Show that the position of the particle relative to $O$ when $t = 4$ is $(38\mathbf{i} + 27\mathbf{j})$.

a) To find the acceleration vector, differentiate each component of the velocity:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(8 + 2t)\mathbf{i} + \frac{d}{dt}(t^3 - 6t)\mathbf{j}$$

$$= \mathbf{2i + (3t^2 - 6)j}$$

b) First find an expression for $\mathbf{s}$ in terms of $t$ by integrating $\mathbf{v}$:

$$\mathbf{s} = \int \mathbf{v}\,dt = \left[\int (8 + 2t)\,dt\right]\mathbf{i} + \left[\int (t^3 - 6t)\,dt\right]\mathbf{j}$$

$$= (8t + t^2)\mathbf{i} + \left(\frac{t^4}{4} - 3t^2\right)\mathbf{j} + \mathbf{C}$$

You still need a constant of integration, but it will be a vector with **i** and **j** components.

When $t = 2$, $\mathbf{s} = (10\mathbf{i} + 3\mathbf{j})$, so use this info to find the vector $\mathbf{C}$:

$$10\mathbf{i} + 3\mathbf{j} = 20\mathbf{i} - 8\mathbf{j} + \mathbf{C}$$

$$\Rightarrow \mathbf{C} = (10 - 20)\mathbf{i} + (3 - -8)\mathbf{j} = -10\mathbf{i} + 11\mathbf{j}$$

Collect **i** and **j** terms and add/subtract to simplify.

$$\text{So } \mathbf{s} = (8t + t^2 - 10)\mathbf{i} + \left(\frac{t^4}{4} - 3t^2 + 11\right)\mathbf{j}$$

When $t = 4$, $\mathbf{s} = (32 + 16 - 10)\mathbf{i} + (64 - 48 + 11)\mathbf{j} = \mathbf{38i + 27j}$ — as required.

## *Practice Questions*

Q1 A particle is travelling at constant velocity $\begin{pmatrix}-3\\2\end{pmatrix}$ ms$^{-1}$. At $t = 0$, the particle has position vector $\begin{pmatrix}12\\-7\end{pmatrix}$ m relative to a fixed origin $O$. Find its position vector at $t = 7$ s.

Q2 A particle has initial velocity $(7\mathbf{i} + 3\mathbf{j})$ ms$^{-1}$ and acceleration $(0.1\mathbf{i} + 0.3\mathbf{j})$ ms$^{-2}$.
Find the speed and direction of the particle after 4 seconds.

Q3 A particle moving in a plane has displacement vector $\mathbf{s}$, where $\mathbf{s} = x\mathbf{i} + y\mathbf{j}$.
What quantities are represented by the vectors $\dot{\mathbf{s}}$ and $\ddot{\mathbf{s}}$?

Q4 A particle sets off from the origin at $t = 0$ and moves in a plane with velocity $\mathbf{v} = (4t\mathbf{i} + t^2\mathbf{j})$ ms$^{-1}$.
Find the displacement vector $\mathbf{s}$ and the acceleration vector $\mathbf{a}$ for the particle at time $t$ seconds.

**Exam Questions**

Q1 A particle $P$ is moving in a horizontal plane with constant acceleration.
After $t$ seconds, $P$ has position vector:

$$[(2t^3 - 7t^2 + 12)\mathbf{i} + (3t^2 - 4t^3 - 7)\mathbf{j}]\text{ m}$$

where the unit vectors **i** and **j** are in the directions of east and north respectively. Find:

a) an expression for the velocity of $P$ after $t$ seconds, [2 marks]

b) the speed of $P$ when $t = 0.5$, and the direction of motion of $P$ at this time. [3 marks]

Q2 A particle is initially at position vector $(\mathbf{i} + 2\mathbf{j})$ m, travelling with constant velocity $(3\mathbf{i} + \mathbf{j})$ ms$^{-1}$.
After 8 s it reaches point $A$. A second particle has constant velocity $(-4\mathbf{i} + 2\mathbf{j})$ and takes 5 s to travel from point $A$ to point $B$. Find the position vectors of points $A$ and $B$. [4 marks]

Q3 A particle moves in a plane with acceleration $\mathbf{a} = (12t\mathbf{i} - e^{\frac{1}{3}t}\mathbf{j})$ ms$^{-2}$. The particle starts at the origin with initial velocity $\mathbf{u} = (5\mathbf{i} - 4\mathbf{j})$ ms$^{-1}$. Find its displacement vector from the origin after 1 second. [5 marks]

## *All this work in two dimensions has left me feeling a bit flat...*

*At least there's not much new to learn on these pages — you're just applying vectors to kinematics. You do need to be comfortable with vector notation (unit and column vectors) and calculus though. After that, it's all fine and dandy.*

# Describing 2D Motion Using Vectors

*I bet you'd forgotten about vectors written with* ***i*** *and* ***j****. Well, they're back. You can use them to write vectors such as displacement, velocity and acceleration in terms of their separate horizontal (****i****) and vertical (****j****) components.*

## *For particles travelling at* ***Constant Velocity, s = vt***

If a particle is travelling at a constant velocity vector, **v**, then its displacement vector, **s**, after time, $t$, can be found using $\mathbf{s} = \mathbf{v}t$.

**Example:** At $t = 0$, a particle has position vector $(6\mathbf{i} + 8\mathbf{j})$ m relative to a fixed origin $O$. The particle is travelling at constant velocity $(2\mathbf{i} - 6\mathbf{j})$ ms$^{-1}$. Find its position vector at $t = 4$ s.

First find its displacement using $\mathbf{s} = \mathbf{v}t$: $\mathbf{s} = 4(2\mathbf{i} - 6\mathbf{j}) = (8\mathbf{i} - 24\mathbf{j})$ m

Then add this to its original position vector:

$(6\mathbf{i} + 8\mathbf{j}) + (8\mathbf{i} - 24\mathbf{j}) = (6 + 8)\mathbf{i} + (8 - 24)\mathbf{j} =$ **$(14\mathbf{i} - 16\mathbf{j})$ m**

Add the **i** and **j** components separately.

## *You might need to use the* ***Constant Acceleration Equations***

If a particle is accelerating at a constant rate, you can use the **constant acceleration equations** from page 186.

**Example:** A particle, $P$, has position vector $\begin{pmatrix}3\\1\end{pmatrix}$ m and velocity $\begin{pmatrix}2\\-5\end{pmatrix}$ ms$^{-1}$ at $t = 0$.
Given that $P$ accelerates at a rate of $\begin{pmatrix}-2\\1\end{pmatrix}$ ms$^{-2}$, find its position vector at $t = 6$ s.

Using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$: $\mathbf{s} = 6\begin{pmatrix}2\\-5\end{pmatrix} + \frac{1}{2}(6^2)\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}12\\-30\end{pmatrix} + \begin{pmatrix}-36\\18\end{pmatrix} = \begin{pmatrix}-24\\-12\end{pmatrix}$ m

So new position vector $= \begin{pmatrix}3\\1\end{pmatrix} + \begin{pmatrix}-24\\-12\end{pmatrix} = \begin{pmatrix}\mathbf{-21}\\\mathbf{-11}\end{pmatrix}$ **m**

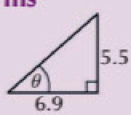
**s**, **u**, **v**, and **a** are all vectors, but t is a scalar. Since vectors can't be squared, you can't use $\mathbf{v}^2 = \mathbf{u}^2 + 2\mathbf{as}$.

**Example:** Find the speed and direction of motion of a particle after 3 s if its initial velocity is $(6\mathbf{i} + 4\mathbf{j})$ ms$^{-1}$ and acceleration is $(0.3\mathbf{i} + 0.5\mathbf{j})$ ms$^{-2}$.

Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$: $\mathbf{v} = (6\mathbf{i} + 4\mathbf{j}) + 3(0.3\mathbf{i} + 0.5\mathbf{j})$
$= (6\mathbf{i} + 4\mathbf{j}) + (0.9\mathbf{i} + 1.5\mathbf{j}) =$ **$(6.9\mathbf{i} + 5.5\mathbf{j})$ ms$^{-1}$**

Speed = magnitude of $\mathbf{v} = \sqrt{6.9^2 + 5.5^2} =$ **8.82 ms$^{-1}$** (3 s.f.)

Direction $= \tan^{-1}\left(\frac{5.5}{6.9}\right) =$ **38.6°** (1 d.p.)




The direction of a velocity vector gives the direction of motion of the object, and the direction of its acceleration vector is the direction of the resultant force (see p.200).

See p.138 for more on the direction of a vector.