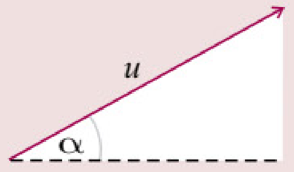
Projectiles and Motion Under Gravity

A 'projectile' is just any old object that's been lobbed through the air. When you're doing projectile questions you'll have to model the motion of particles in two dimensions, usually ignoring air resistance.

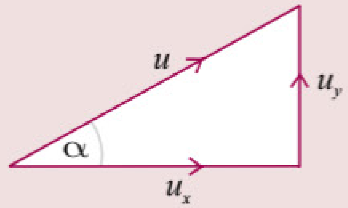
Split ***Velocity of Projection*** *into* ***Two Components***

A particle projected with a speed u at an angle α to the horizontal has **two components** of initial velocity — one **horizontal** (parallel to the x-axis) and one **vertical** (parallel to the y-axis).
These are called **x and y components**, and they make projectile questions dead easy to deal with:

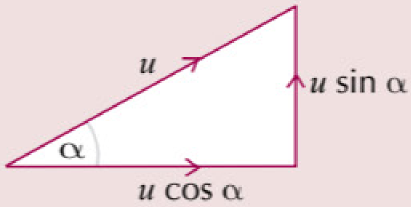
Here's the same information in a diagram:



Split the velocity into its x and y components:



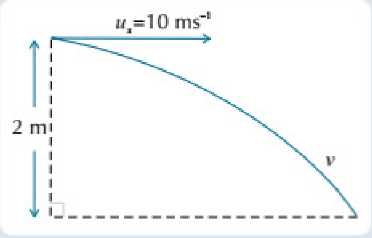
Finally, work out the **values** of the components using **trigonometry**:



Split the ***Motion*** *into* ***Horizontal*** *and* ***Vertical Components*** *too*

Split the motion into horizontal and vertical components. Then deal with them separately using the **suvat equations**.
The only thing that's the same in both directions is **time** — so this connects the two directions.
For projectile questions, the only acceleration is **vertical** and due to **gravity** — **horizontal acceleration is zero.**

Example: a) A stone is thrown horizontally with speed 10 ms^{-1} from a height of 2 m above the horizontal ground. Find the time taken for the stone to hit the ground and the horizontal distance travelled before impact. Use $g = 9.8$ ms^{-2}.



Vertical motion
(take down as +ve):

$s = 2$ $\quad u = u_y = 0$

$a = 9.8$ $\quad t = ?$

The stone only has velocity in the x-direction.

$s = ut + \frac{1}{2}at^2$

$2 = (0 \times t) + \left(\frac{1}{2} \times 9.8 \times t^2\right)$

$t = 0.6388... =$ **0.639 s** (3 s.f.)

i.e. the stone lands after 0.639 seconds.

Horizontal motion
(take right as +ve):

$s = ?$ $\quad u = u_x = 10$

$a = 0$ $\quad t = 0.6388...$

The same as for the vertical motion.

$s = ut + \frac{1}{2}at^2$

$= (10 \times 0.6388...) + \left(\frac{1}{2} \times 0 \times 0.6388...^2\right)$

$=$ **6.39 m** (3 s.f.)

i.e. the stone has travelled 6.39 m horizontally when it lands.

b) Find the speed and direction of the stone after 0.5 s.

Again, keep the vertical and horizontal bits separate:

Vertical motion $\quad v = u + at$

$v_y = 0 + 9.8 \times 0.5$

$=$ **4.9 ms^{-1}**

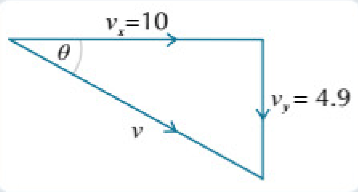
Horizontal motion $\quad v = u + at$

$v_x = 10 + 0 \times 0.5$

$=$ **10 ms^{-1}**

v_x is always equal to u_x when there's no horizontal acceleration.

Now you can find the speed and direction...



Speed $= |v| = \sqrt{4.9^2 + 10^2} =$ **11.1 ms^{-1}** (3 s.f.)

$\tan\theta = \frac{4.9}{10} \Rightarrow \theta = \tan^{-1}\left(\frac{4.9}{10}\right) = 26.1°$ (1 d.p.)

So the direction of the stone's motion is 360° – 26.1° = **331.9°** (1 d.p.)

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Velocity = 0 at the Maximum Height

Example: A cricket ball is projected with a speed of 30 ms^{-1} at an angle of 25° above the horizontal. Assume the ground is horizontal and the ball is struck from a point 1.5 m above the ground. Find:
a) the maximum height above the ground the ball reaches (h),
b) the horizontal distance travelled by the ball before it hits the ground (r),
c) the length of time the ball is at least 5 m above the ground.

a) **Vertical motion up to the maximum height** (take up as +ve):

$s = ?$ $u = 30 \sin 25°$
$v = 0$ $a = -9.8$

The ball will momentarily stop moving vertically when it reaches its maximum height.

$v^2 = u^2 + 2as$
$0 = (30 \sin 25°)^2 + 2(-9.8 \times s)$
$s = 8.20130...$ m
$h = 8.20130... + 1.5 =$ **9.70 m** (3 s.f.)

Don't forget to add the height from which the ball is hit.

b) **Vertical motion until ball hits ground** (take up as +ve):

$s = -1.5$
$u = 30 \sin 25°$
$a = -9.8$
$t = ?$

$s = ut + \frac{1}{2}at^2$

$-1.5 = (30 \sin 25°)t - \frac{1}{2}(9.8)t^2$

$t^2 - 2.58745...t - 0.30612... = 0$

$t = -0.11334...$ or $t = 2.70080...$

Using the quadratic formula you get two answers, but time can't be negative, so forget about this answer.

So $t =$ **2.70 s** (3 s.f.)

Horizontal motion (take right as +ve)

$s = r$ $u = 30 \cos 25°$
$a = 0$ $t = 2.70080...$

$s = ut + \frac{1}{2}at^2$

$r = 30 \cos 25° (2.70080...) + \frac{1}{2}(0)(2.70080...)^2$

$r =$ **73.4 m** (3 s.f.)

c) **Vertical motion whilst ball is at least 5 m above ground** (take up as +ve):

$s = 3.5$
$u = 30 \sin 25°$
$a = -9.8$
$t = ?$

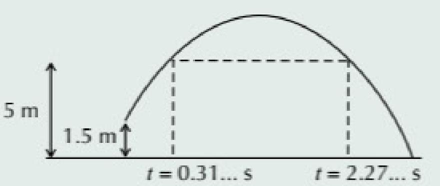
The ball is hit from 1.5 m above ground, so 5 m − 1.5 m = 3.5 m

$s = ut + \frac{1}{2}at^2$

$3.5 = (30 \sin 25°)t - \frac{1}{2}(9.8)t^2$

$t^2 - (2.58745...)t + (0.71428...) = 0$

Using the quadratic formula:
$t = 0.31421...$ or $t = 2.27324...$



These are the two times when the ball is 5 m above the ground.

So, length of time at least 5 m above the ground:
$2.27324... - 0.311421... =$ **1.96 s** (3 s.f.)

The Components of Velocity can be described using i and j Vectors

Your old friends, **i** and **j** vectors, are pretty useful for describing projectiles.

Example: A stone is thrown from a point 1.2 metres above the horizontal ground. It travels for 4 seconds then lands on the ground. The stone is thrown with velocity $(2q\mathbf{i} + q\mathbf{j})$ ms^{-1}, where **i** and **j** are the horizontal and vertical unit vectors respectively. Find the value of q and the initial speed of the stone.

Vertical motion in j direction (take up as +ve):

$s = -1.2$ $u = q$
$a = -9.8$ $t = 4$

$s = ut + \frac{1}{2}at^2$

$-1.2 = 4q - \frac{1}{2}(9.8)4^2$

$4q = 77.2$, so **$q = 19.3$**

Now find the **initial speed** of the stone:

speed $= \sqrt{(2q)^2 + q^2}$
$= \sqrt{38.6^2 + 19.3^2} =$ **43.2 ms^{-1}** (3 s.f.)

The vertical component of velocity is q, and the horizontal component is $2q$.

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You can **Derive General Formulas** for the motion of projectiles

Just one last example of projectile motion. But boy is it a beauty...

Example: a) A golf ball is struck from a point A on a horizontal plane.
When the ball has moved a horizontal distance x, its height above the plane is y.
The ball is modelled as a particle projected with initial speed u ms^{-1} at an angle α.

Show that $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$.

This is the general formula for the path of a projectile.

The formula includes motion in both directions (x and y), so form two equations and substitute one into the other:

Horizontal motion (taking right as +ve):

$s = x$ $\quad u_x = u\cos\alpha$

$a = 0$ $\quad t = t$

Using $s = ut + \frac{1}{2}at^2$:

When you're using these variables, this is the obvious equation to use.

$x = u\cos\alpha \times t$

Rearrange to make t the subject:

$t = \dfrac{x}{u\cos\alpha}$ — call this equation ①

t doesn't appear in the final formula, so by making it the subject you can eliminate it.

Vertical motion (taking up as +ve):

$s = y$ $\quad u_y = u\sin\alpha$

$a = -g$ $\quad t = t$

Using $s = ut + \frac{1}{2}at^2$:

It would be a massive pain to make t the subject here, so do it with the other equation.

$y = (u\sin\alpha \times t) - \frac{1}{2}gt^2$ — call this equation ②

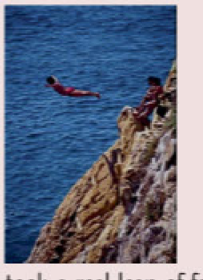
t is the same horizontally and vertically, so you can **substitute** ① into ② and eliminate t:

$$y = u\sin\alpha \times \frac{x}{u\cos\alpha} - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\frac{\sin\alpha}{\cos\alpha} - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\alpha}\right)$$

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

$$\mathbf{y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}}$$ — as required.



Dave took a real leap of faith with his projectile calculations...

b) The ball just passes over the top of a 10 m tall tree, which is 45 m away. Given that $\alpha = 45°$, find the speed of the ball as it passes over the tree.

Substitute $x = 45$, $y = 10$ and $\alpha = 45°$ into the result above and rearrange to find the speed of projection u:

$$10 = 45\tan 45° - \frac{9.8 \times 45^2}{2u^2 \times \cos^2 45°} = 45 - \frac{19\,845}{u^2}$$

$35u^2 = 19\,845 \Rightarrow$ **$u = 23.81176...$ ms^{-1}**

Avoid rounding if possible — your calculator's memory can be useful here.

Then find the components of the ball's velocity as it passes over the tree:

Horizontal motion (taking right as +ve):

$v_x = u_x = 23.81176...\cos 45° =$ **16.83745... ms^{-1}**

Remember — with projectiles there's no horizontal acceleration, so v_x always equals u_x.

Vertical motion (taking up as +ve):

$s = 10$ $\quad u_y = 23.81176...\sin 45°$

$v_y = v_y$ $\quad a = -g$

Using $v^2 = u^2 + 2as$:

$v_y^2 = 283.5 - 2 \times 9.8 \times 10 =$ **87.5**

Don't bother finding the square root, as you need v_y^2 in the final step. Sneaky.

Now you can find the speed: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{16.83745...^2 + 87.5}$

$v = 19.3$ ms^{-1} (to 3 s.f.)

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Three whole pages of examples about projectiles calls for a full page of questions.
Don't worry — you'll be thanking me for all the practice come exam time.

Practice Questions

Q1 A rifle fires a bullet horizontally at 120 ms^{-1}. The target is hit at a horizontal distance of 60 m from the end of the rifle. Find how far the target is vertically below the end of the rifle. Take $g = 9.8$ ms^{-2}.

Q2 A golf ball takes 4 seconds to land after being hit with a golf club from a point on the horizontal ground. If it leaves the club with velocity $(2q\mathbf{i} + 3q\mathbf{j})$ ms^{-1}, find its initial horizontal velocity. Take $g = 9.8$ ms^{-2}.

Q3 A particle is projected with initial velocity u ms^{-1} at an angle α above the horizontal.

a) Find, in terms of u and α, the horizontal and vertical components of the particle's initial velocity.

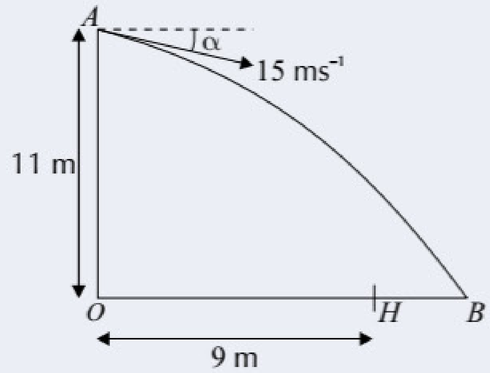
b) (i) Show that the particle reaches its maximum height after $\frac{u\sin\alpha}{g}$ seconds.

(ii) Hence, and using a suitable trig identity, show that the horizontal range is $\frac{u^2\sin(2\alpha)}{g}$ metres.

c) Show that the maximum vertical height the particle reaches is $\frac{u^2\sin^2\alpha}{2g}$ metres.

Exam Questions

Q1



A stone is thrown from point A on the edge of a cliff, towards a point H, which is on horizontal ground. The point O is on the ground, 11 m vertically below the point of projection. The stone is thrown with speed 15 ms^{-1} at an angle α below the horizontal, where $\tan\alpha = \frac{3}{4}$. The horizontal distance from O to H is 9 m. The stone misses the point H and hits the ground at point B, as shown above. Find:

a) the time taken by the stone to reach the ground, [5 marks]

b) the horizontal distance the stone misses H by, [3 marks]

c) the speed of projection which would have ensured that the stone landed at H. [5 marks]

Q2 A stationary football is kicked with a speed of 20 ms^{-1}, at an angle of 30° to the horizontal, towards a goal 30 m away. The crossbar is 2.5 m above the level ground. Assuming the path of the ball is not impeded, determine whether the ball passes above or below the crossbar, stating all modelling assumptions. [6 marks]

Q3 A golf ball is hit from a tee at point O on the edge of a vertical cliff. Point O is 30 m vertically above A, the base of the cliff. The ball is hit with velocity $(14\mathbf{i} + 35\mathbf{j})$ ms^{-1} towards a hole, H, which lies on the horizontal ground. At time t seconds, the position of the ball is $(x\mathbf{i} + y\mathbf{j})$ m, relative to O. **i** and **j** are the horizontal and vertical unit vectors respectively. Take $g = 9.8$ ms^{-2}.

a) By writing down expressions for x and y in terms of t, show that $y = \frac{5x}{2} - \frac{x^2}{40}$. [4 marks]

The ball lands on the ground at point B, 7 m beyond H, where AHB is a straight horizontal line.

b) Find the horizontal distance AB. [3 marks]

c) Find the speed of the ball as it passes through a point vertically above H. [4 marks]

Projectiles — they're all about throwing up. Or across. Or slightly down...

You've used the equations of motion before and there isn't much different here. Projectile questions can be wordy so it can help to draw a diagram of the situation before doing anything else. The main thing to remember is that horizontal acceleration is zero — great news because it makes half the calculations as easy as... something very easy.