Friction and Inclined Planes

Friction tries to prevent motion, but don't let it prevent you getting marks in the exam. It's like my gran always says — revision is the best way to keep things running smoothly and avoid chafing. Revision, and a dash of talcum powder.

Friction tries to Prevent Motion

Push hard enough and a particle will move, even though there's friction opposing the motion. A **frictional** (**contact**) **force**, F, has a **maximum value**. This depends on the **roughness** of the surface and the value of the **normal reaction** (or **normal contact force**) from the surface.

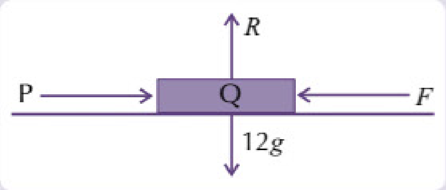
$F \le \mu R$ OR $F \le \mu N$ (where R and N both stand for normal reaction)

μ has no units. You pronounce it 'mew'.

Try one of our refreshing smoothies — low on calories, low on μ, and one of your 5-a-day!

μ is called the "**coefficient of friction**". The **rougher** the surface, the **bigger** μ gets.

Example: What range of values can a frictional force take to resist a horizontal force P acting on a particle Q, of mass 12 kg, resting on a rough horizontal plane which has a coefficient of friction of 0.4? Take $g = 9.8$ ms^{-2}.



Resolving vertically: $R = 12g$

Use formula from above: $F \le \mu R$

$F \le 0.4(12g)$

$F \le$ **47.04 N**

The **resultant** of the **frictional** contact force and the **normal** contact force is often just called the **contact force** between the object and the surface.

So friction can take any value between 0 and 47.04 N, depending on how large P is. If P < 47.04 N then Q remains in equilibrium. If P = 47.04 N then Q is **on the point of sliding** — i.e. friction is at its **limit**. If P > 47.04 then Q will start to move.

Limiting Friction is when friction is at Maximum (F = μR)

In **static** equilibrium, F ≤ μR, and in **limiting** equilibrium (or when **moving**), F = μR.

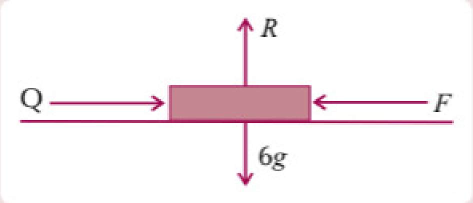
Example: A particle of mass 6 kg is placed on a rough horizontal plane which has a coefficient of friction of 0.3. A horizontal force Q is applied to the particle. Describe what happens if Q is: a) 16 N b) 20 N
Take $g = 9.8$ ms^{-2}

Resolving vertically: $R = 6g$

Using formula above: $F \le \mu R$

$F \le 0.3(6g)$

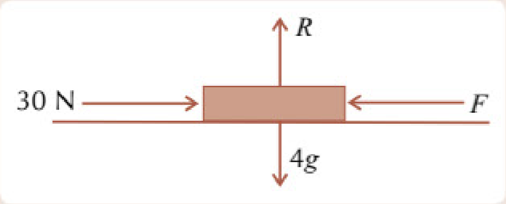
$F \le$ **17.64 N**



a) Since Q < 17.64 N, it **won't move**.

b) Since Q > 17.64 N, it'll **start moving**. No probs.

Example: A particle of mass 4 kg at rest on a rough horizontal plane is being pushed by a horizontal force of 30 N. Given that the particle is on the point of moving, find the coefficient of friction.



Resolving horizontally: $F = 30$

Resolving vertically: $R = 4g$

The particle's about to move, so friction is at its limit:

$F = \mu R$

$30 = \mu(4g)$

$\mu = \frac{30}{4g} = $ **0.77** (2 d.p.)

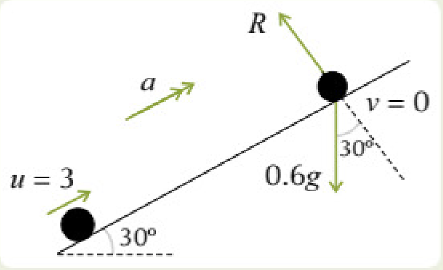
μ is usually given to 2 d.p.

Friction and Inclined Planes

Use $F = ma$ in **Two Directions** for **Inclined Plane** questions

For **inclined slope** questions, it's much easier to resolve forces **parallel** and **perpendicular** to the plane's surface. Here's a nice example without friction to ease you in gently...

Example: A mass of 600 g is propelled up the line of greatest slope of a smooth plane inclined at 30° to the horizontal. If its velocity is 3 ms^{-1} after the propelling force has stopped, find the distance it travels before v reaches 0 ms^{-1} and the magnitude of the normal reaction. Use $g = 9.8$ ms^{-2}.



Resolve parallel to the plane (↗):

$F_{net} = ma$

$-0.6g \sin 30° = 0.6a$

$\Rightarrow a = -4.9$ ms^{-2}

Watch the units — m = 600 g = 0.6 kg

$v^2 = u^2 + 2as$

$0 = 3^2 + 2(-4.9)s$

So $s =$ **0.918 m** (3 s.f.)

Resolve perpendicular to plane (↖):

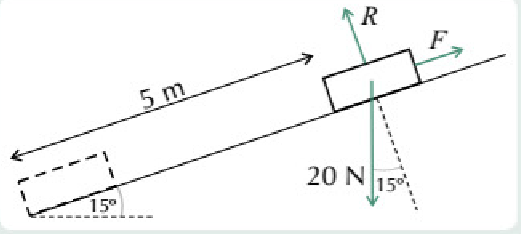
$F_{net} = ma$

$R - 0.6g \cos 30° = 0.6 \times 0$

So $R =$ **5.09 N** (3 s.f.)

Remember that friction always acts in the **opposite** direction to the motion.

Example: A small body of weight 20 N accelerates from rest and moves a distance of 5 m down a rough plane angled at 15° to the horizontal. Draw a force diagram and find the coefficient of friction between the body and the plane given that the motion takes 6 seconds. Take $g = 9.8$ ms^{-2}.



$s = 5, \quad u = 0, \quad a = a, \quad t = 6$

Use one of the equations of motion:

$$s = ut + \frac{1}{2}at^2$$

$$5 = (0 \times 6) + \left(\frac{1}{2}a \times 6^2\right)$$

so $a =$ **0.2777... ms^{-2}**

Resolving perpendicular (↖):

$F_{net} = ma$

$R - 20 \cos 15° = \frac{20}{g} \times 0$

$R = 20 \cos 15° =$ **19.31... N**

Resolving parallel (↙):

$F_{net} = ma$

$20 \sin 15° - F = \frac{20}{g} \times 0.2777...$

$F =$ **4.609... N**

It's sliding, so $F = \mu R$

$4.609... = \mu \times 19.31...$

$\mu =$ **0.24** (2 d.p.)

Friction opposes motion, so it also increases the **tension** in whatever's doing the pulling...

Example: A mass of 3 kg is being pulled up a plane inclined at 20° to the horizontal by a rope parallel to the surface. Given that the mass is accelerating at 0.6 ms^{-2} and that the coefficient of friction is 0.4, find the tension in the rope. Take $g = 9.8$ ms^{-2}.

Resolving perpendicular (↖):

$F_{net} = ma$

$R - 3g \cos 20° = 3 \times 0$

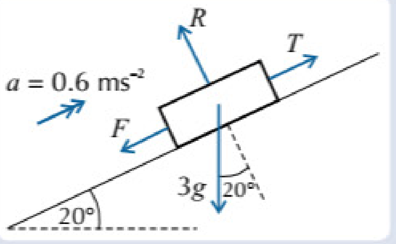
$R = 3g \cos 20° =$ **27.62... N**

The mass is sliding:

$F = \mu R$

$= 0.4 \times 27.62...$

$=$ **11.05... N**



Resolving parallel (↙):

$F_{net} = ma$

$T - F - 3g \sin 20° = 3 \times 0.6$

$T = 1.8 + 11.05... + 3g \sin 20° =$ **22.9 N** (3 s.f.)

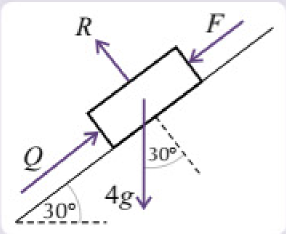
Friction and Inclined Planes

Friction Opposes Limiting Motion

For a body **at rest** but on the point of moving **down** a plane, the friction force is **up** the plane. A body about to move **up** a plane is opposed by friction **down** the plane. Remember it well (it's about to come in handy).

Example: A 4 kg box is placed on a 30° plane where $\mu = 0.4$. A force Q maintains equilibrium by acting up the plane parallel to the line of greatest slope. Find Q if the box is on the point of sliding: a) up the plane, b) down the plane.

a)



Resolving perpendicular (↖):

$F_{net} = ma$

$R - 4g\cos 30° = 0$

$R = 4g\cos 30°$

$F = \mu R$

$= 0.4 \times 4g\cos 30°$

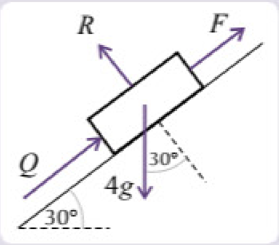
$F = 1.6g\cos 30°$

Resolving parallel (↗):

$Q - 4g\sin 30° - F = 4 \times 0$

$Q = 4g\sin 30° + 1.6g\cos 30° =$ **33.2 N** (3 s.f.)

b)



Resolving perpendicular (↖):

$F_{net} = ma$

$R - 4g\cos 30° = 0$

$R = 4g\cos 30°$

$F = \mu R$

$= 0.4 \times 4g\cos 30°$

$F = 1.6g\cos 30°$

The maximum value of F is the same in both cases.

Resolving parallel (↗):

$Q - 4g\sin 30° + F = 4 \times 0$

$Q = 4g\sin 30° - 1.6g\cos 30° =$ **6.02 N** (3 s.f.)

So for equilibrium **$6.02 \text{ N} \le Q \le 33.2 \text{ N}$**

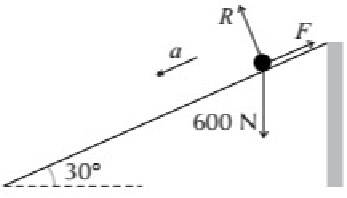
Practice Questions

Q1 a) Describe the motion of a mass of 12 kg pushed by a force of 50 N parallel to the rough horizontal plane on which the mass is placed. The plane has coefficient of friction $\mu = 0.5$.

b) What minimum force would be needed to move the mass in part a)?

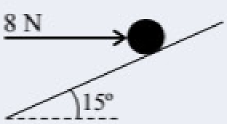
Q2 A brick of mass 1.2 kg is sliding down a rough plane which is inclined at 25° to the horizontal. Given that its acceleration is 0.3 ms^{-2}, find the coefficient of friction between the brick and the plane. What assumptions have you made?

Q3 An army recruit of weight 600 N steps off a tower and accelerates down a "death slide" wire as shown. The recruit hangs from a light rope held between her hands and looped over the wire. The coefficient of friction between the rope and wire is 0.5. Given that the wire is 20 m long and makes an angle of 30° to the horizontal throughout its length, find how fast the recruit is travelling when she reaches the end of the wire.



Exam Questions

Q1 A horizontal force of 8 N just stops a mass of 7 kg from sliding down a plane inclined at 15° to the horizontal, as shown.



a) Calculate the coefficient of friction between the mass and the plane to 2 d.p. [5 marks]

b) The 8 N force is now removed. Find how long the mass takes to slide a distance of 3 m down the line of greatest slope. [7 marks]

Q2 A 10 kg box is being held in equilibrium on a plane inclined at 30° to the horizontal by a force P acting parallel to the plane. Given that the coefficient of friction between the box and the plane is 0.4, find the range of possible values for the magnitude of P. [7 marks]

Sometimes friction really rubs me up the wrong way...

Friction can be a right nuisance, but without it we'd just slide all over the place, which would be worse (I imagine).

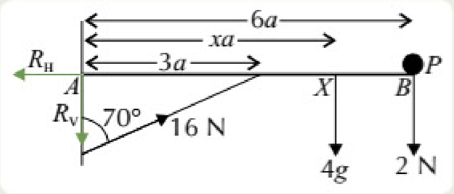
Rigid Bodies and Friction

Where would we be without friction? Well, for one thing, using a ladder would certainly be trickier...

Reactions *can have* ***Horizontal and Vertical Components***

If a rod is connected to a plane (such as a wall) by a **hinge** or **pivot**, and the forces holding it in equilibrium **aren't parallel**, the reaction at the wall **won't be perpendicular** to the wall. Resolving into components is helpful here.

Example: A non-uniform rod, AB, of length $6a$ m and mass 4 kg is held in equilibrium by a light strut at an angle of 70° to a vertical wall, as shown. The distance from A to X, the centre of mass of the rod, is xa m. The strut exerts a thrust of 16 N at the centre of the rod. A particle of weight 2 N is placed at B. Find x, and the magnitude and direction of the reaction at A.



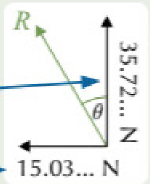
Put the unknown reaction at the wall as a horizontal and a vertical component. Don't worry about their directions — you'll just get a negative answer if you're wrong.

Taking moments about A:
$(4g \times xa) + (2 \times 6a) = 16 \cos 70° \times 3a$
$4gx = 4.416... \Rightarrow$ **$x = 0.113$** (3 s.f.)

You're looking at the forces on the rod, so there's no reaction force upwards at B.

Resolving vertically:
$R_V + 4g + 2 = 16 \cos 70°$
so $R_V = 16 \cos 70° - 39.2 - 2$
$\Rightarrow R_V = -35.72...$ N (so R_V acts upwards)

Resolving horizontally:
$R_H = 16 \sin 70° \Rightarrow R_H = 15.03...$ N



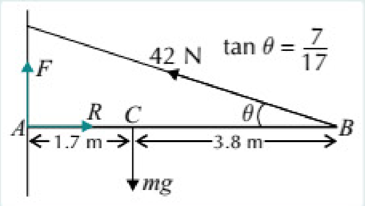
Magnitude of reaction:
$|R| = \sqrt{R_H^2 + R_V^2} = \sqrt{15.03...^2 + 35.72...^2}$
$= $ **38.8 N** (3 s.f.)

Direction of reaction:
$\tan \theta = \frac{15.03...}{35.72...}$
$\Rightarrow$ **$\theta = 22.8°$** (3 s.f.) to the wall

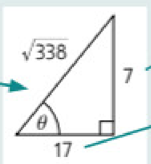
Friction *lets you assume the* ***Reaction is Perpendicular***

A rod **attached** to a wall has a **reaction** at the wall with a **horizontal** and **vertical** component.
If the rod is held by **friction** instead, then the frictional force 'replaces' the **vertical** component.

Example: A rod, AB, rests against a rough vertical wall and is held in limiting equilibrium perpendicular to the wall by a light inextensible string attached at B at an angle of θ, as shown, where $\tan \theta = \frac{7}{17}$. The tension in the string is 42 N. The length AB is 5.5 m and the centre of mass is located 3.8 m from B. Find the mass of the rod, m, and the coefficient of friction, μ, between the wall and the rod.



First, work out what $\sin \theta$ and $\cos \theta$ are by drawing a right-angled triangle:



$\sin \theta = \frac{7}{\sqrt{338}} = 0.3807...$

$\cos \theta = \frac{17}{\sqrt{338}} = 0.9246...$

Taking moments about A means you can find mg while ignoring the unknowns F and R.

Moments about A: $mg \times 1.7 = 42 \sin \theta \times 5.5$
$mg = 51.73...$ N
$m = 5.28$ kg (3 s.f.)

Now take moments about a different point to find F. I've taken them about C, but you could have used B.

Moments about C: $1.7 \times F = 3.8 \times 42 \sin \theta$
$F = 35.74...$ N

Now you know F, you only need to find R before you can find μ.

Resolving horizontally: $R = 42 \cos \theta$
$R = 38.83...$ N

Limiting equilibrium showed up on p.205 — it means that the body is on the point of moving.

Equilibrium is limiting, so: $F = \mu R$
$35.74... = 38.83... \mu$
$\mu = 0.92$ (2 d.p.)

Rigid Bodies and Friction

Multiple Surfaces *can exert a* ***Frictional Force***

When a rod rests at an angle against two surfaces, **either** or **both** of them could exert a frictional force. 'Ladder' questions are a common example of this — keep an eye out for when the floor or the wall are **rough**.

Example: A ladder, modelled as a uniform rod of mass 1.3 kg and length $5x$ m, rests against a smooth wall at an angle of 65° to the rough ground, as shown. A cat of mass 4.5 kg sits on the ladder at C, $4x$ m from the base.
Given that the ladder is in limiting equilibrium, find the coefficient of friction between the ground and the ladder.

Take moments about the base of the ladder to find N:

$N \sin 65° \times 5x = (1.3g \cos 65° \times 2.5x) + (4.5g \cos 65° \times 4x)$

$4.531...xN = 13.46...x + 74.59...x$

$N = \frac{88.01...x}{4.531...x} = 19.42...$ N

Resolve vertically to find R:

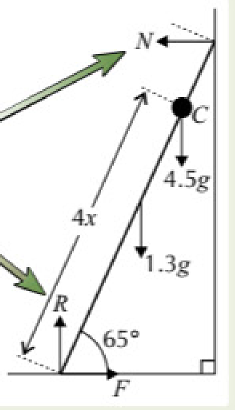
$R = 1.3g + 4.5g = 56.84$ N

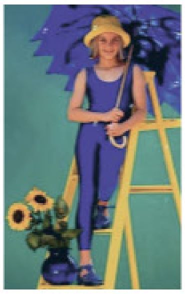
The ladder is in limiting equilibrium, so $F = \mu R$:

Resolving horizontally shows $F = N$, so $19.42... = 56.84\mu$

$\mu = \mathbf{0.34}$ (2 d.p.)

R is the normal reaction of the ground and N is the normal reaction of the wall.

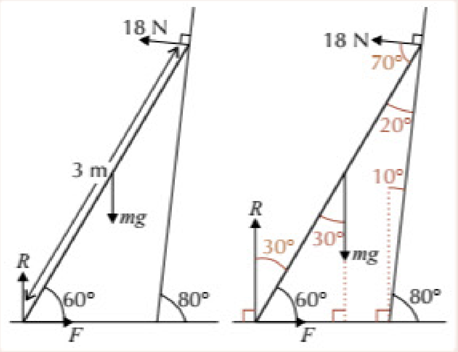




No ladder problems here...

Sometimes a body may be leaning against a surface that **isn't vertical** (this blows my mind). Just remember that the **reaction force** always acts **perpendicular to the surface**.

Example: A uniform ladder of length 3 m rests against a smooth wall slanted at 80° to the horizontal, as shown. The ladder is at an angle of 60° to the ground. The magnitude of the normal reaction of the wall is 18 N. Find the mass of the ladder.



The diagram on the right shows that the angle that the reaction force from the wall makes with the ladder is 70°.

Taking moments about the base of the ladder:

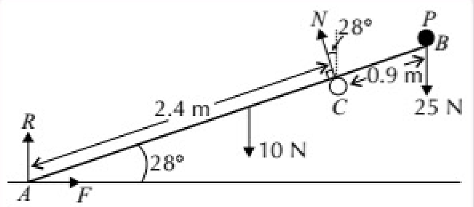
$mg \sin 30° \times 1.5 = 18 \sin 70° \times 3$

so $m = 50.74... \div 7.35 =$ **6.90 kg** (3 s.f.)

Bodies can be Supported ***Along Their Lengths***

If a rod is **resting** on something along its length then the reaction is **perpendicular** to the **rod**.

Example: A uniform rod, AB, rests with end A on rough ground and upon a smooth peg at C, as shown. A particle, P, with weight 25 N is placed at B. Given that the rod is in limiting equilibrium, find the normal reaction, N, at the peg and the friction, F, between the rod and the ground.



Moments about A:

$2.4N = (10 \cos 28° \times [(2.4 + 0.9) \div 2]) + (25 \cos 28° \times 3.3)$

$\Rightarrow$ $\mathbf{N = 36.4}$ **N** (3 s.f.)

Resolving horizontally:

$F = N \sin 28° = 36.4 \sin 28°$

$\Rightarrow$ $\mathbf{F = 17.1}$ **N** (3 s.f.)

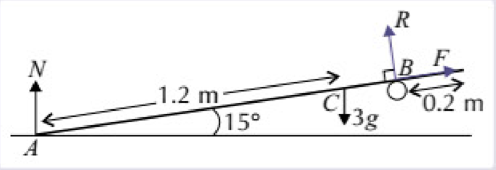
It's a uniform rod, so its weight acts in the middle.

Rigid Bodies and Friction

You can work out a **Range** for μ

In limiting equilibrium, friction is at its **maximum** (i.e. $F = \mu R$). You might be asked to find μ when you **don't know** if equilibrium is limiting. Find it in the same way as if equilibrium was limiting, but replace $F = \mu R$ with $F \leq \mu R$.

Example: A rough peg supports a rod of length 1.5 m at a point B, as shown. It rests on a smooth horizontal plane at A, and its weight acts at point C. Given that the friction at the peg exerts a force of 8 N, show that $\mu \geq 0.31$ to 2 d.p.



You know F, but to find μ you also need to know R.

Take moments about A to find R:

$R \times (1.5 - 0.2) = 3g \cos 15° \times 1.2$

$R = 26.2...$ N

$F \leq \mu R$,

$\mu \geq 8 \div 26.2...$

$\mu \geq$ **0.31** (2 d.p.) as required

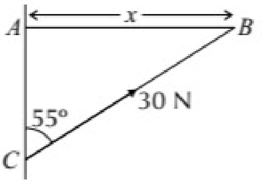
Practice Questions

Q1 A uniform ladder, of length l m, is placed on rough horizontal ground and rests against a smooth vertical wall at an angle of 20° to the wall. Draw a diagram modelling this system with forces labelled. State the assumptions you would make.

Q2 A non-uniform rod, AB, is freely hinged at a vertical wall. It is held horizontally in equilibrium by a strut attached to the wall at C at an angle of 55°, as shown. The thrust in the strut is 30 N. The rod is x m long, and has mass 2 kg, centred 0.4 m from A. Taking $g = 9.8$ ms^{-2}, find:

a) the length of the rod, x,

b) the magnitude and direction of the reaction at A.



Exam Questions

Whenever a numerical value of g is required in the following questions, take $g = 9.8$ ms^{-2}.

Q1 A uniform rod of mass m kg rests in equilibrium against rough horizontal ground at point A and a smooth peg at point B, making an angle of θ with the ground, where $\sin \theta = \frac{3}{5}$. The rod is l m long and B is $\frac{3}{4}l$ from A.

a) Show that the normal reaction at the peg, $N = \frac{2}{3}mg \cos \theta$. [3 marks]

b) Find the range of possible values of the coefficient of friction between the rod and the ground. [6 marks]

Q2 A uniform ladder AB rests in limiting equilibrium against a smooth vertical wall (A) and upon the rough horizontal ground (B) at an angle of θ. Clive stands on the ladder at point C a third of the way along its length from the base B. The ladder is 4.2 m long and weighs 180 N. The normal reaction at A is 490 N. Given that $\tan \theta = \frac{8}{11}$, find:

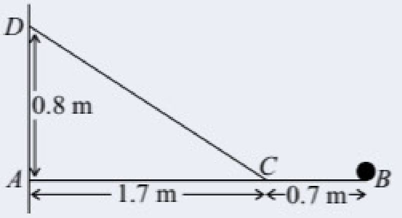
a) the mass of Clive, m, to the nearest kg, [3 marks]

b) the coefficient of friction, μ, between the ground and the ladder. [4 marks]

Q3 A uniform rod, AB, of mass 3 kg is held horizontally in limiting equilibrium against a rough wall by an inextensible string connected to the rod at point C and the wall at point D, as shown. A particle of mass m kg rests at point B. The magnitude of the normal reaction of the wall at A is 72.5 N. Find:

a) the tension, T, in the string, and the mass, m, of the particle, [5 marks]

b) the frictional force, F, between the wall and the rod. [2 marks]



...and that's everything you need to know about anything, ever.

If you've been revising the topics in order, this is the end of A-level Maths (and the end of my quips, sadly). Of course, there's no harm in going back for another go. Have a nice cup of tea first though — you deserve it. There are also some exam tips coming up that are definitely worth checking out. You'll be a master of maths before you know it.