Summary of key points

1 Linear simultaneous equations can be solved using elimination or substitution.

2 Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.

3 The solutions of a pair of simultaneous equations represent the points of intersection of their graphs.

4 For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:

- $b^2 - 4ac > 0$ two real solutions
- $b^2 - 4ac = 0$ one real solution
- $b^2 - 4ac < 0$ no real solutions

5 The solution of an inequality is the set of all real numbers x that make the inequality true.

6 To solve a quadratic inequality:

- Rearrange so that the right-hand side of the inequality is 0
- Solve the corresponding quadratic equation to find the critical values
- Sketch the graph of the quadratic function
- Use your sketch to find the required set of values.

7 The values of x for which the curve $y = \mathrm{f}(x)$ is **below** the curve $y = \mathrm{g}(x)$ satisfy the inequality $\mathrm{f}(x) < \mathrm{g}(x)$.

The values of x for which the curve $y = \mathrm{f}(x)$ is **above** the curve $y = \mathrm{g}(x)$ satisfy the inequality $\mathrm{f}(x) > \mathrm{g}(x)$.

8 $y < \mathrm{f}(x)$ represents the points on the coordinate grid below the curve $y = \mathrm{f}(x)$.

$y > \mathrm{f}(x)$ represents the points on the coordinate grid above the curve $y = \mathrm{f}(x)$.

9 If $y > \mathrm{f}(x)$ or $y < \mathrm{f}(x)$ then the curve $y = \mathrm{f}(x)$ is not included in the region and is represented by a dotted line.

If $y \geqslant \mathrm{f}(x)$ or $y \leqslant \mathrm{f}(x)$ then the curve $y = \mathrm{f}(x)$ is included in the region and is represented by a solid line.

Simultaneous equations

If a pair of simultaneous equations involves an x^2 or a y^2 term, you need to solve them using **substitution**. Remember to **number** the equations to keep track of your working.

$y = x^2 - 2x - 7$ ①

$x - y = -3$ ②

Rearrange the linear equation to make y the subject.

From ②: $y = x + 3$ ③

$x + 3 = x^2 - 2x - 7$

Substitute $x + 3$ for y in equation ①.

$0 = x^2 - 3x - 10$

$0 = (x - 5)(x + 2)$

$x = 5$ or $x = -2$

Each solution for x has a corresponding value of y. Substitute to find the values of y.

The solutions are $x = 5$, $y = 8$ and $x = -2$, $y = 1$.

Worked example

Solve the simultaneous equations

$x - 2y = 1$ ①

$x^2 + y^2 = 13$ ② **(6 marks)**

From ①: $x = 1 + 2y$ ③

Substitute $1 + 2y$ for x in ②:

$(1 + 2y)^2 + y^2 = 13$

$1 + 4y + 4y^2 + y^2 = 13$

$5y^2 + 4y - 12 = 0$

$(5y - 6)(y + 2) = 0$

$y = \frac{6}{5}$ or $y = -2$

$x = 1 + 2\left(\frac{6}{5}\right) = \frac{17}{5}$ $x = 1 + 2(-2) = -3$

Solutions: $x = \frac{17}{5}$, $y = \frac{6}{5}$ and $x = -3$, $y = -2$

You can substitute for x or y. It is easier to substitute for x because there will be no fractions.

Use brackets to make sure that the whole expression is squared.

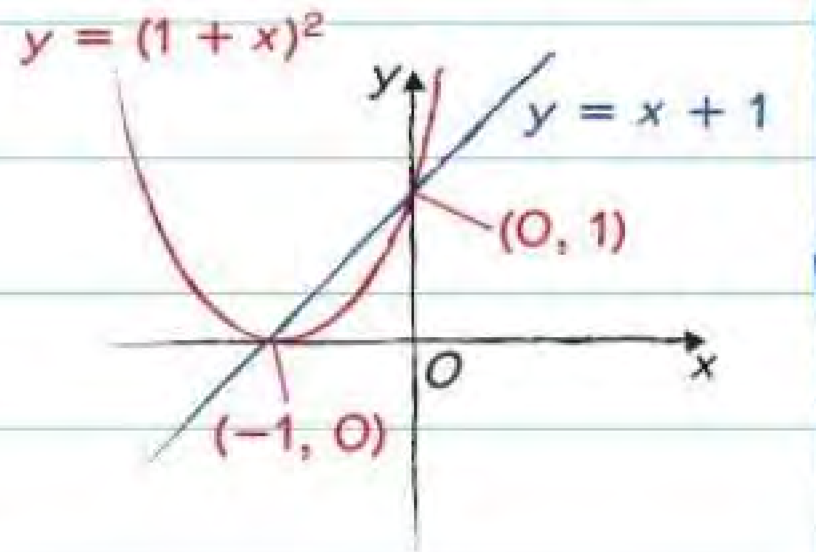
Rearrange the quadratic equation for y into the form $ay^2 + by + c = 0$

Factorise the left-hand side to find two solutions for y.

Remember that there will be **two pairs** of solutions. Each value of y will produce a corresponding value of x. You need to find **four** different values in total **and** pair them up correctly.

Thinking graphically

The solutions to a pair of simultaneous equations correspond to the points where the graphs of the equations **intersect**.
Because an equation involving x^2 or y^2 represents a **curve**, there can be more than one point of intersection. Each point has an x-value and a y-value. You can write the solutions using coordinates.



There is more on intersections of graphs on page 16.

Now try this

1 Solve the simultaneous equations

$x + y = 5$

$x^2 + 2y^2 = 22$ **(6 marks)**

Answer

You will need to write $x^2 - 6x + 7$ in completed square form. Have a look at page 4 for a reminder.

2 (a) By eliminating y from the simultaneous equations

$y = x + 6$

$xy - 2x^2 = 7$

show that $x^2 - 6x + 7 = 0$ **(2 marks)**

Answer

(b) Hence solve the simultaneous equations in part (a), giving your answers in the form $a \pm \sqrt{2}$, where a is an integer. **(4 marks)**

Answer

Inequalities

You might need to find a set of values which **satisfy** an inequality. If the inequality involves a **quadratic** expression you should always **sketch a graph** to help you answer the question.

This graph shows a sketch of $y = (x - 1)(x + 5)$.

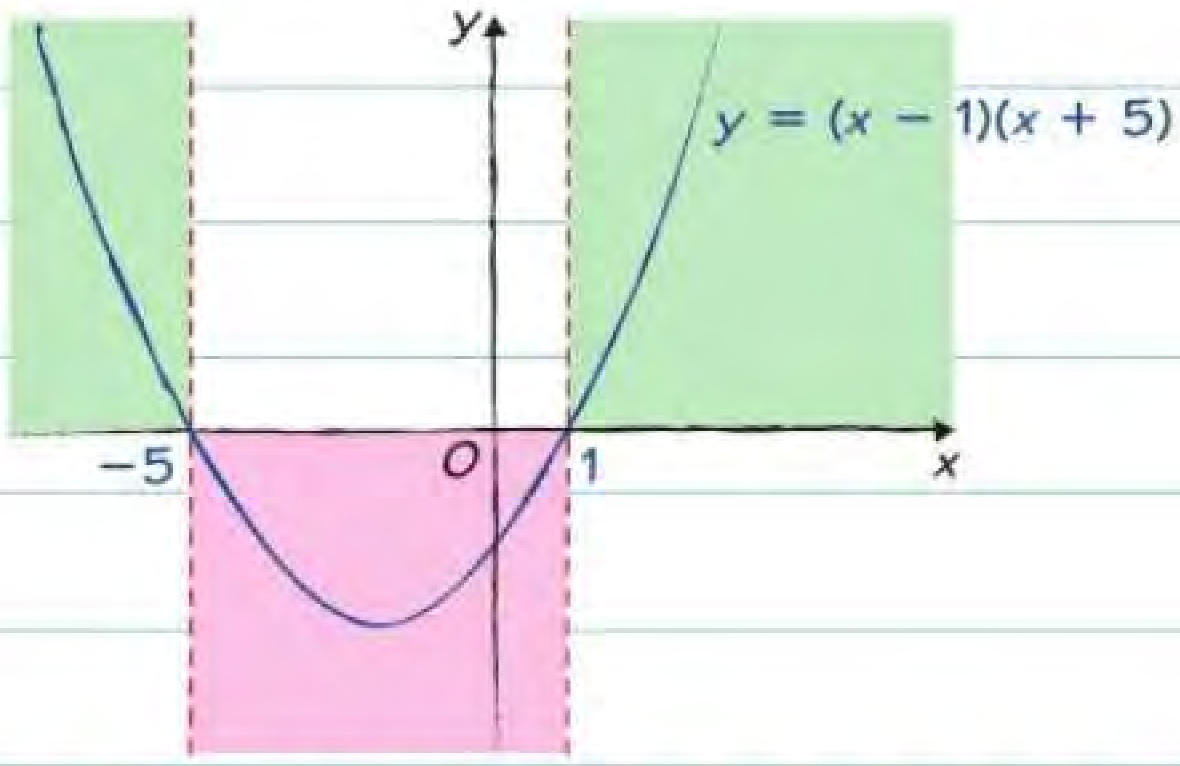
(pink) For these values of x, the curve is **below** the x-axis, so $y < 0$.

The solution to $(x - 1)(x + 5) < 0$ is $-5 < x < 1$

(green) For these values of x, the curve is **above** the x-axis, so $y > 0$.

The solution to $(x - 1)(x + 5) > 0$ is $x < -5$ or $x > 1$

There are two separate sets of values which satisfy this inequality. You need to give both sets of values, and write OR between them.



Worked example

Find the set of values for which

(a) $8x - 7 < 5x + 5$ **(2 marks)**

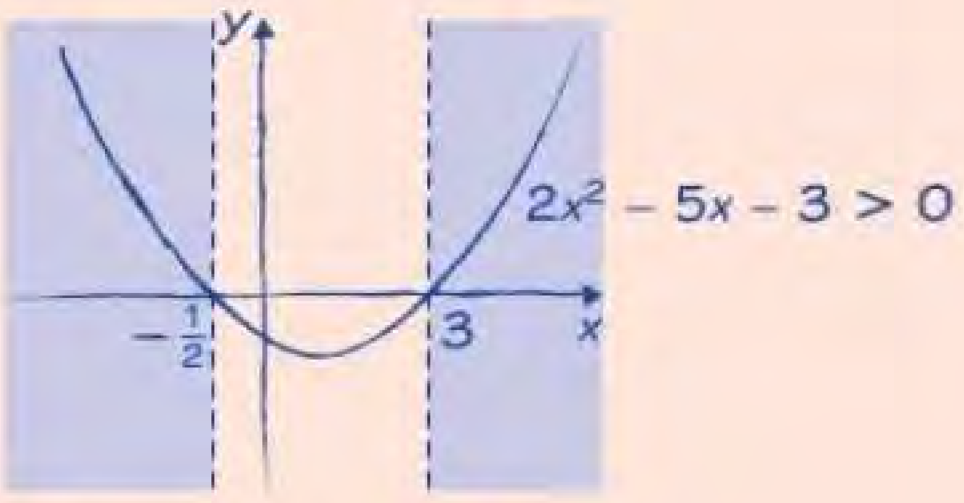
$8x < 5x + 12$

$3x < 12$

$x < 4$

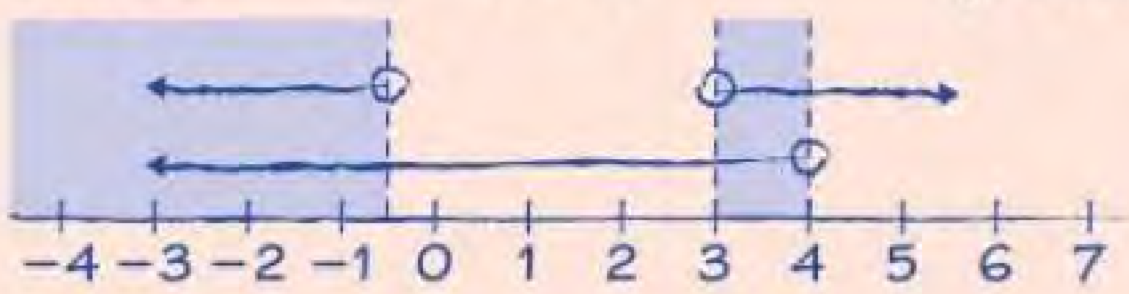
(b) $2x^2 - 5x - 3 > 0$ **(4 marks)**

$(2x + 1)(x - 3) > 0$



$x < -\frac{1}{2}$ or $x > 3$

(c) **both** $8x - 7 < 5x + 5$ and $2x^2 - 5x - 3 > 0$ **(3 marks)**



$x < -\frac{1}{2}$ or $3 < x < 4$

Quadratic inequalities

Follow these steps to solve any quadratic inequality:

1. Rearrange so one side is 0.
2. Factorise the other side.
3. Sketch the graph.
4. Write the solutions using $<$, $>$, $\leqslant$ or $\geqslant$.

Make sure you write **two separate** inequalities for your answer. You can't write $3 < x < -\frac{1}{2}$. It's not true, because 3 is larger than $-\frac{1}{2}$.

Problem solved!

You need **both** inequalities to be true **at the same time**. Draw them both on a number line and look for the values where they **overlap**.

You will need to use problem-solving skills throughout your exam – **be prepared!**

You can also write this as $\{x : x < -\frac{1}{2}\} \cup \{x : 3 < x < 4\}$

Now try this

1 Find the set of values of x for which $x(x - 5) < 14$ **(4 marks)**

Answer

You need to expand the brackets and rearrange the inequality into the form $ax^2 + bx + c < 0$ first.

2 The equation $x^2 + (k - 3)x - 4k$ has two distinct real roots.

(a) Show that k satisfies $k^2 + 10k + 9 > 0$ **(3 marks)**

(b) Hence find the set of possible values for k. **(4 marks)**

Answer

Answer

Inequalities on graphs

You can interpret inequalities graphically by considering curves and regions. This diagram shows the graphs of $y = x^2 - 4x$ and $y = -2x + 3$:

The graphs intersect when:

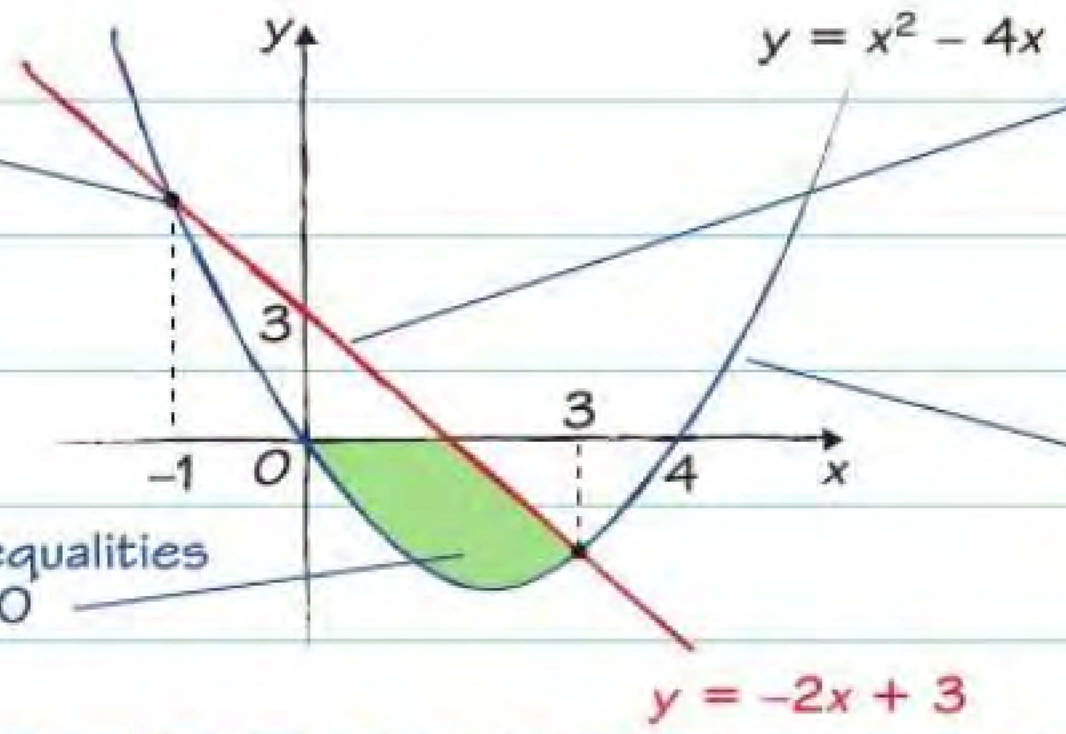
$x^2 - 4x = -2x + 3$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

There is more on intersections of graphs on page 16.



The shaded **region** satisfies the inequalities $y \leqslant -2x + 3$, $y \geqslant x^2 - 4x$ and $y \leqslant 0$ simultaneously.

Between the points of intersection the line is **above** the curve. So the solution to $-2x + 3 > x^2 - 4x$ is $-1 < x < 3$.

Outside the points of intersection the line is **below** the curve. So the solution to $-2x + 3 < x^2 - 4x$ is $x < -1$ or $x > 3$.

Worked example

The graph shows the line with equation $y = x - 6$ and the curve with equation $y = 5x - 2x^2$.

(a) Determine the coordinates of the points of intersection of the line and the curve. **(4 marks)**

$x - 6 = 5x - 2x^2$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

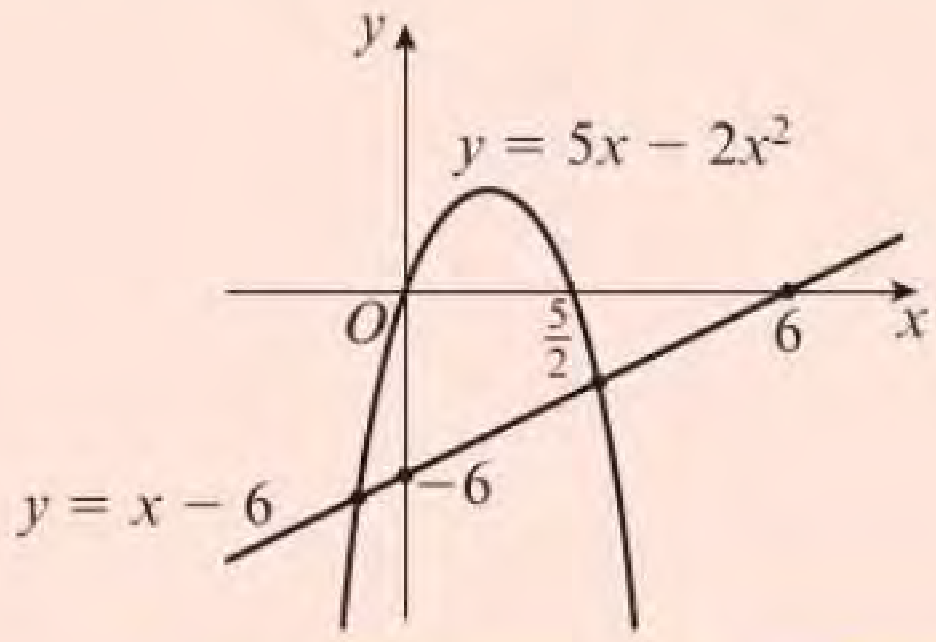
$x = -1$ or $x = 3$

When $x = -1$, $y = -1 - 6 = -7$. P_1 is $(-1, -7)$

When $x = 3$, $y = 3 - 6 = -3$. P_2 is $(3, -3)$

(b) Hence solve the inequality $x - 6 > 5x - 2x^2$. **(1 mark)**

$\{x : x < -1\} \cup \{x : x > 3\}$



The solutions are the x-values when the line is higher than the curve. You can use set notation or give your answer as '$x < -1$ or $x > 3$'.

Worked example

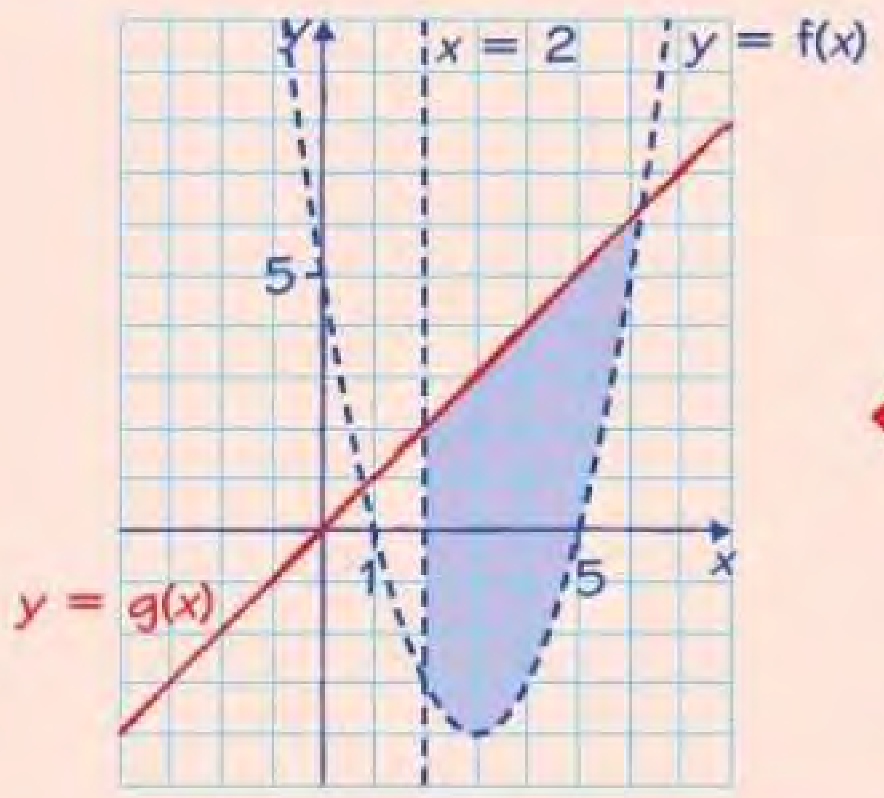
$f(x) = x^2 - 6x + 5$

$g(x) = x$

On a graph, show the region that satisfies the inequalities:

$y > f(x)$ $\quad y \leqslant g(x)$ $\quad x > 2$

(5 marks)



Use dotted lines to show a strict inequality (< or >) and solid lines to show a non-strict inequality (≤ or ≥).

Now try this

$f(x) = 6 - x^2 \qquad g(x) = \frac{1}{2}x + 1$

(a) On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$. Indicate any points of intersection with the coordinate axes. Answer **(3 marks)**

(b) Determine the x-coordinates of any points of intersection of the two graphs. **(3 marks)**

Answer

(c) Hence solve the inequalities Answer

(i) $f(x) < g(x)$ (ii) $f(x) \geqslant g(x)$ **(2 marks)**

(d) Shade the region on the graph that satisfies all of the following inequalities:

$y \leqslant f(x)$, $y \geqslant g(x)$, $x \geqslant 0$ **(1 mark)**

Answer