Summary of key points

1 When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.

2 You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.

3 The **factor theorem** states that if $f(x)$ is a polynomial then:

- If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$
- If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$

Expanding and factorising

In your AS exam, you might have to **multiply out** (or **expand**) a product of **three** brackets, or **factorise** a **cubic** expression.

Expanding brackets

To expand the product of two factors, you have to multiply **every term** in the first factor by **every term** in the second factor.

There are 2 terms in the first factor and 3 terms in the second factor, so there will be $2 \times 3 = 6$ terms in the expanded expression **before** you collect like terms.

$2x \times$

$$(2x + 3)(5x^2 - x + 4) = 10x^3 - 2x^2 + 8x + 15x^2 - 3x + 12$$
$$= 10x^3 + 13x^2 + 5x + 12$$

$3 \times$

Simplify your expression by collecting like terms: $-2x^2 + 15x^2 = 13x^2$

Worked example

Show that $(3 + 2\sqrt{x})^2$ can be written as $9 + k\sqrt{x} + 4x$, where k is a constant to be found. **(2 marks)**

$$(3 + 2\sqrt{x})(3 + 2\sqrt{x}) = 9 + 6\sqrt{x} + 6\sqrt{x} + 4x$$
$$= 9 + 12\sqrt{x} + 4x$$

$k = 12$

Remember that $\sqrt{x} \times \sqrt{x} = x$.
$(2\sqrt{x})^2 = 2\sqrt{x} \times 2\sqrt{x} = 4\sqrt{x}\sqrt{x} = 4x$

Problem solved!

If you have to find a constant, it's a good idea to write down the value of the constant when you have finished your working.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Factorising

Factorising is the opposite of expanding brackets.

Expanding brackets

$$(x + 4)^2 = x^2 + 8x + 16$$
$$(2x + 3)(x - 10) = 2x^2 - 7x - 30$$

Factorising

Worked example

Factorise completely $x^3 + x^2 - 6x$ **(3 marks)**

$$x^3 + x^2 - 6x = x(x^2 + x - 6)$$
$$= x(x + 3)(x - 2)$$

Start by taking the common factor, x, out of every term. You are left with a quadratic expression, which you can factorise into two linear factors.

Now try this

1 Given that $(x + 2)(x + 1)^2 = x^3 + bx^2 + cx + d$, where b, c and d are constants, find the values of b, c and d. Answer **(3 marks)**

2 Factorise completely $3x^3 - 2x^2 - x$ **(3 marks)** Answer

3 Factorise completely $25x^2 - 16$ **(2 marks)**

Answer

Multiply out $(x + 1)^2$ first to get $(x + 2)(x^2 + 2x + 1)$.

This is a difference of two squares: $a^2 - b^2 = (a + b)(a - b)$

The factor theorem

In your exam, you might need to use the factor theorem to help you **factorise polynomials** like these:

$f(x) = 2x^3 + 5x^2 - 15x + 10$

This is a CUBIC because the highest power of x is 3.

$f(x) = x^4 - 4x^3 + 2x^2 - 3$

This is a QUARTIC because the highest power of x is 4.

The factor theorem

If $f(x)$ is a polynomial and $f(p) = 0$, then $(x - p)$ is a **factor** of $f(x)$.

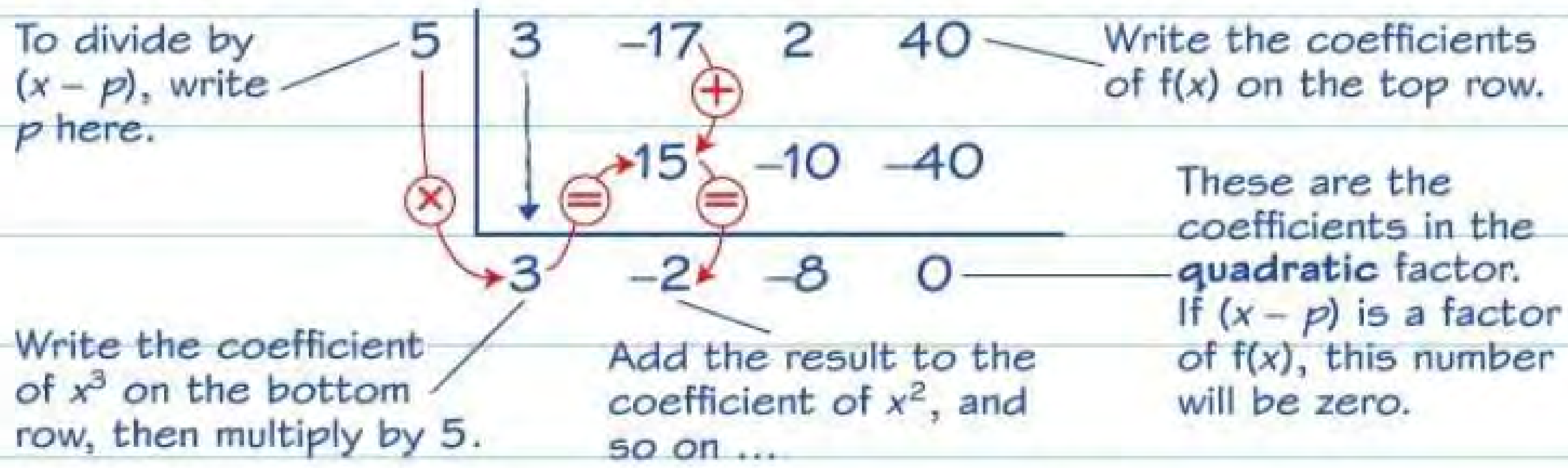
- ✓ Only use this theorem with **polynomials**.
- ✓ Watch out for the **sign**. If $f(-1) = 0$ then the factor would be $(x + 1)$.
- ✓ **Learn** this rule – it's not in the formulae booklet.

Synthetic division with polynomials

If you have to **completely factorise** a **cubic** polynomial, you will usually need to find **three linear factors**. You can find the first one using the factor theorem. When you take this factor out, your other factor will be a **quadratic expression**. One quick way to find this is by using **synthetic division**.

To completely factorise $f(x) = 3x^3 - 17x^2 + 2x + 40$:

1. Use the factor theorem to find one factor: $f(5) = 0$, so $(x - 5)$ is a factor.
2. Use synthetic division to divide $3x^3 - 17x^2 + 2x + 40$ by $(x - 5)$:



3. The quadratic factor is $3x^2 - 2x - 8$. Factorise this to complete the factorisation.
4. So $3x^3 - 17x^2 + 2x + 40 = (x - 5)(3x^2 - 2x - 8) = (x - 5)(3x + 4)(x - 2)$.

Worked example

$f(x) = 2x^3 - x^2 - 15x + 18$

(a) Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$. **(2 marks)**

$f(-3) = 2(-3)^3 - (-3)^2 - 15(-3) + 18$
$= -54 - 9 + 45 + 18 = 0$

So $(x + 3)$ is a factor.

Be careful with the sign (+ or –). The factor theorem says that $(x - p)$ is a factor if $f(p) = 0$, so you need to evaluate $f(-3)$. You need to write down that $(x + 3)$ is a factor at the end of your working.

(b) Factorise $f(x)$ completely. **(4 marks)**

Using synthetic division to divide $f(x)$ by $(x + 3)$:

–3	2	–1	–15	18
		–6	21	–18
	2	–7	6	0

So $\dfrac{2x^3 - x^2 - 15x + 18}{x + 3} = 2x^2 - 7x + 6$

$f(x) = (x + 3)(2x^2 - 7x + 6) = (x + 3)(2x - 3)(x - 2)$

You know that $(x + 3)$ is a factor, so divide $f(x)$ by $(x + 3)$ using **synthetic division**. You could check your answer by expanding the brackets:

$(x + 3)(2x^2 - 7x + 6)$
$= 2x^3 - 7x^2 + 6x + 6x^2 - 21x + 18$
$= 2x^3 - x^2 - 15x + 18$ ✓

You need to factorise $(2x^2 - 7x + 6)$ in the normal way. Look at page 2 for a reminder.

Now try this

1 (a) Use the factor theorem to show that $(x - 2)$ is a factor of $x^3 - 7x^2 - 14x + 48$ Answer **(2 marks)**

(b) Factorise $x^3 - 7x^2 - 14x + 48$ completely. Answer **(4 marks)**

2 $f(x) = 2x^3 - 3x^2 - 65x - a$

(a) Given that $(x + 4)$ is a factor of $f(x)$, find the value of a. Answer **(3 marks)**

(b) Factorise $f(x)$ completely. **(4 marks)** Answer