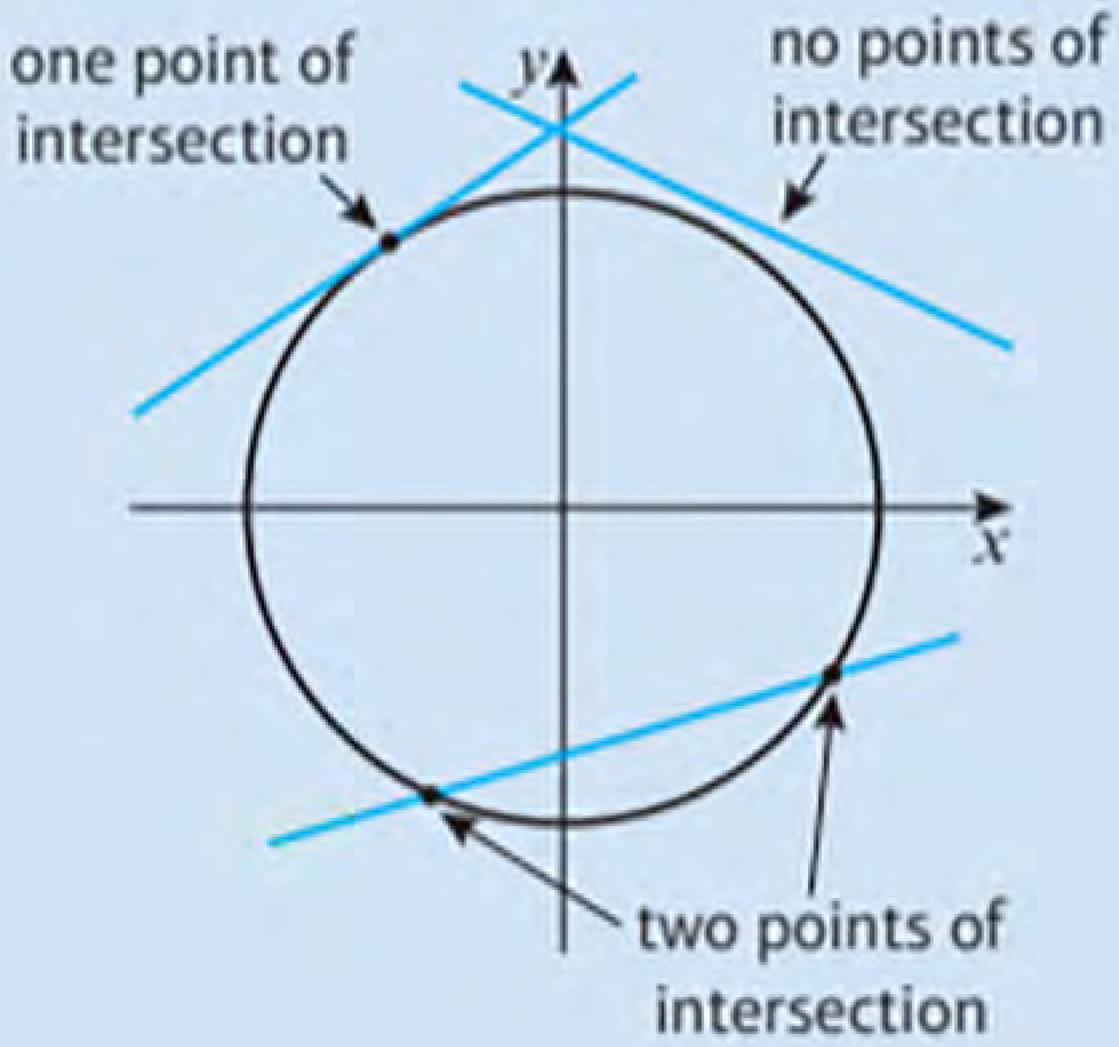
Summary of key points

3 The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

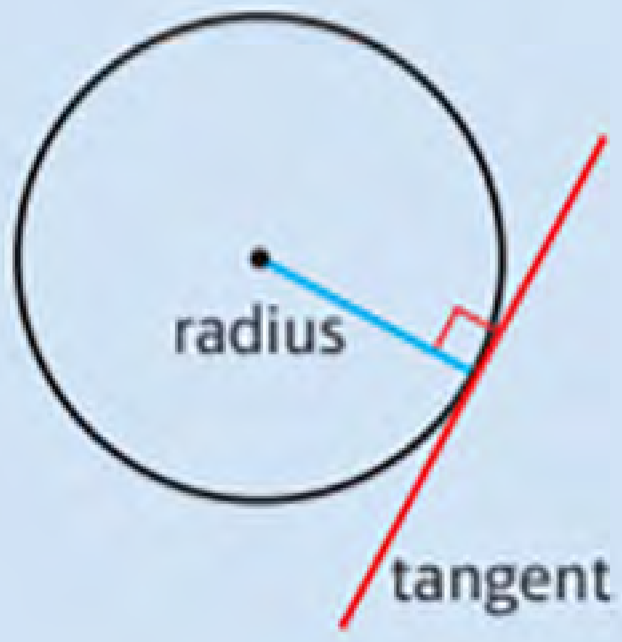
4 The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

5 The equation of a circle can be given in the form: $x^2 + y^2 + 2fx + 2gy + c = 0$
This circle has centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$

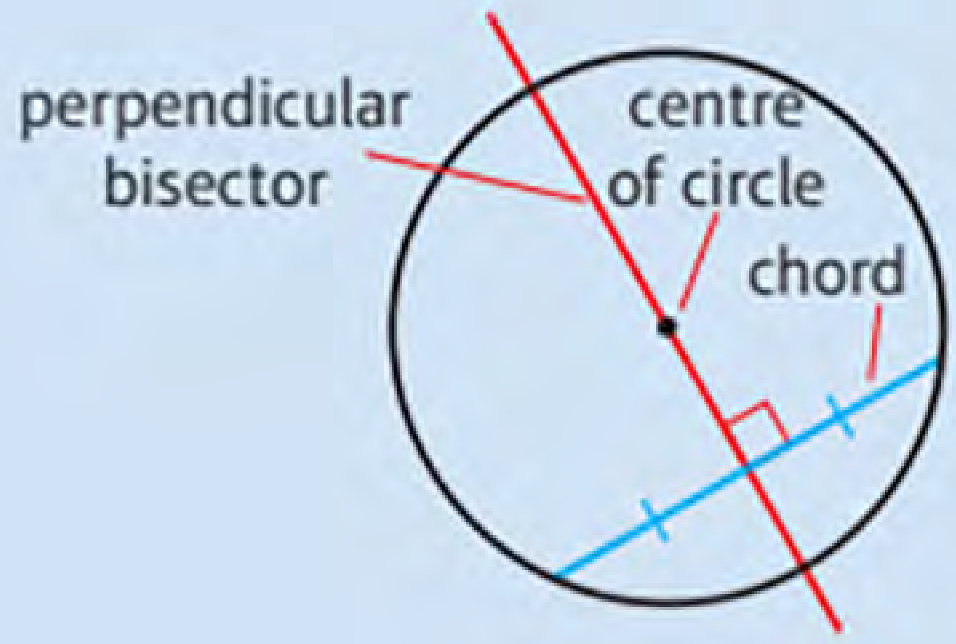
6 A straight line can intersect a circle once, by just touching the circle, or twice. Not all straight lines will intersect a given circle.



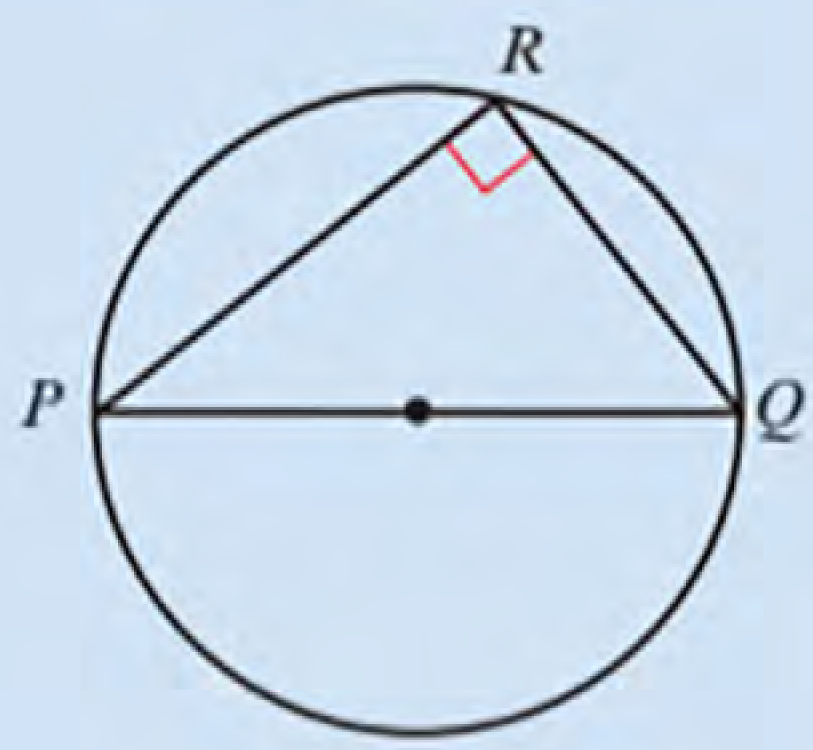
7 A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.



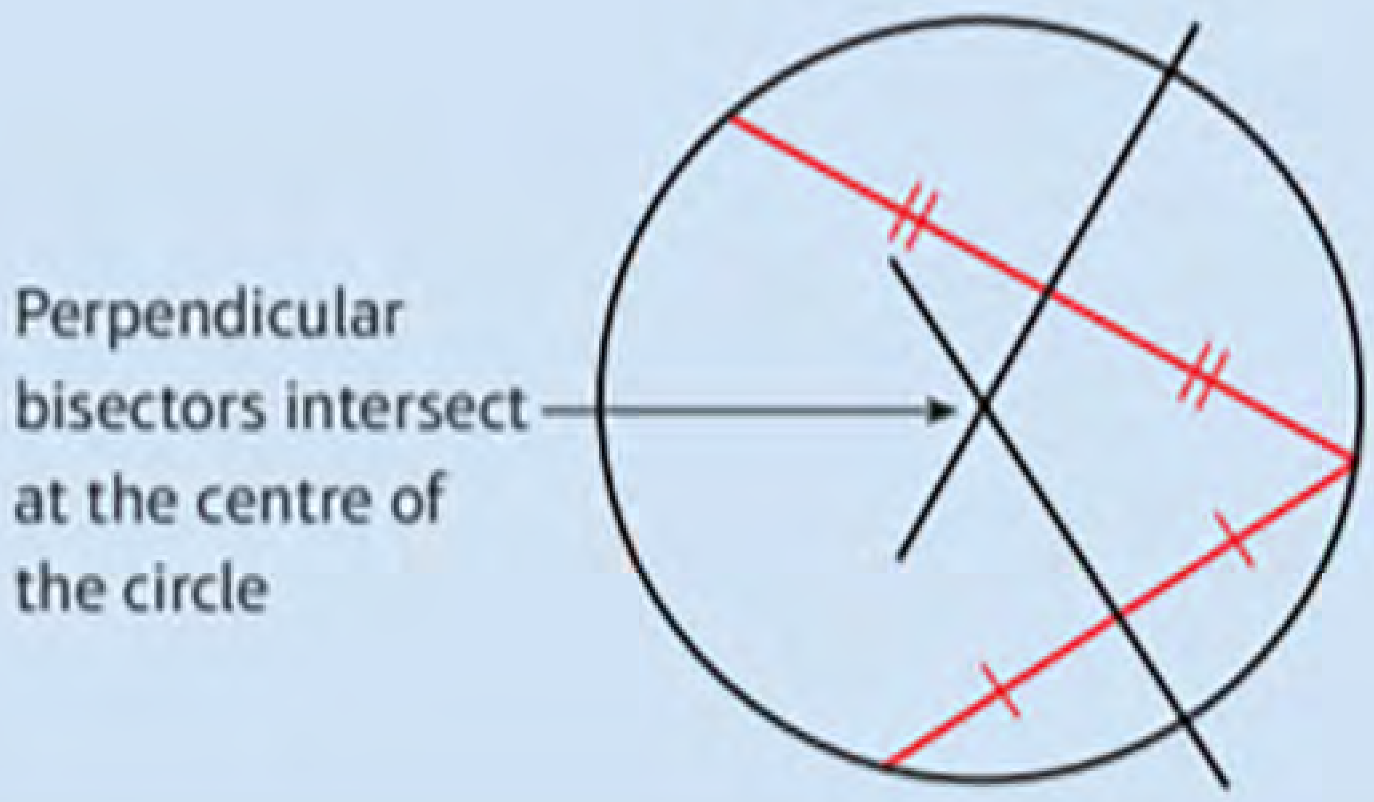
8 The perpendicular bisector of a chord will go through the centre of a circle.



9
- If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.
- The angle in a semicircle is always a right angle.



10 To find the centre of a circle given any three points:
- Find the equations of the perpendicular bisectors of two different chords.
- Find the coordinates of intersection of the perpendicular bisectors.

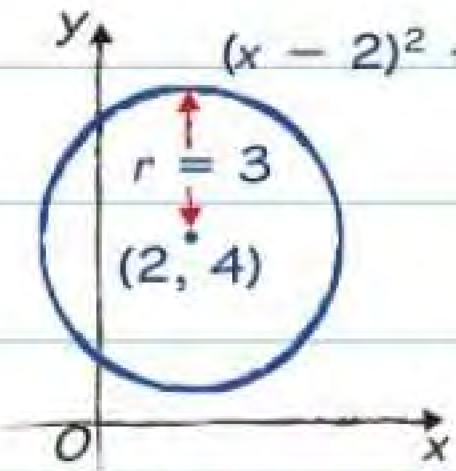


Equation of a circle

A circle with centre (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Be really careful with the right-hand side. It is the radius **squared**.



The formulae booklet does not contain **any** coordinate geometry formulae! You will need to learn them for your exam.

Worked example

A circle C has centre (3, 1) and passes through the point (–2, 5).

(a) Find an equation for C. **(4 marks)**

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(-2 - 3)^2 + (5 - 1)^2} = \sqrt{41}$

$(x - a)^2 + (y - b)^2 = r^2$

$(x - 3)^2 + (y - 1)^2 = 41$

(b) Verify that (7, –4) also lies on C. **(1 mark)**

$(7 - 3)^2 + (-4 - 1)^2 = 4^2 + (-5)^2 = 41$

So point (7, –4) lies on C.

The radius of the circle, r, is the length of the line segment between (3, 1) and (–2, 5). For a reminder about finding the length of a line segment have a look at page 19.

Just because you can use a calculator in your exam, it doesn't mean you always should!

Don't write your radius as a decimal. If you leave it in the form $\sqrt{41}$ it will be exact when you square it to write your equation.

Problem solved!

You need to rearrange the equation into the form $(x - a)^2 + (x - b)^2 = r^2$

This is a bit like **completing the square**. Have a look at page 4 for a recap.

$x^2 - 6x = (x - 3)^2 - 3^2$

$y^2 + 2y = (y + 1)^2 - 1^2$

Remember that the right-hand side of the equation is r^2, so the radius is 5, not 25.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Substitute $x = 0$ into the equation to find the points where C crosses the y-axis.

Worked example

The circle C has equation $x^2 + y^2 - 6x + 2y - 15 = 0$

(a) Find the coordinates of the centre of C and the radius of C. **(4 marks)**

$x^2 - 6x + y^2 + 2y - 15 = 0$

$(x - 3)^2 - 3^2 + (y + 1)^2 - 1^2 - 15 = 0$

$(x - 3)^2 + (y + 1)^2 - 25 = 0$

$(x - 3)^2 + (y + 1)^2 = 5^2$

C has centre (3, –1) and radius 5.

(b) Find the coordinates of the points where C crosses the y-axis. **(2 marks)**

When $x = 0$, $(0 - 3)^2 + (y + 1)^2 = 25$

$(y + 1)^2 = 16$

$y + 1 = \pm 4$

$y = -5$ or 3

C crosses the y-axis at (0, –5) and (0, 3).

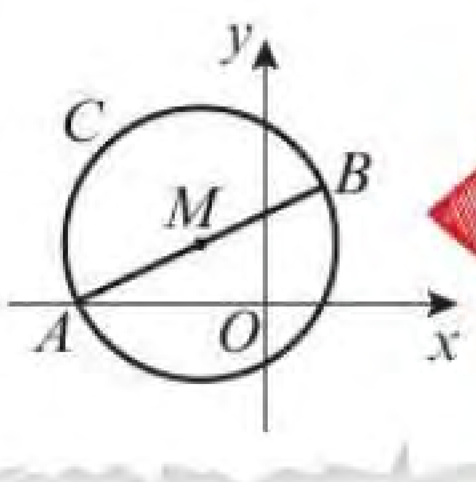
Now try this

$A(-6, 0)$ and $B(2, 4)$ are the endpoints of a diameter of the circle C. Find

(a) the length of AB Answer **(2 marks)**

(b) the coordinates of the midpoint of AB Answer **(2 marks)**

(c) an equation for the circle C. **(2 marks)** Answer



You will need these two key **diameter** facts to solve this problem:

1. The **midpoint** of any diameter of a circle is the centre of the circle.
2. The diameter is **twice** the radius.

Have a look at page 18 for a reminder about midpoints.

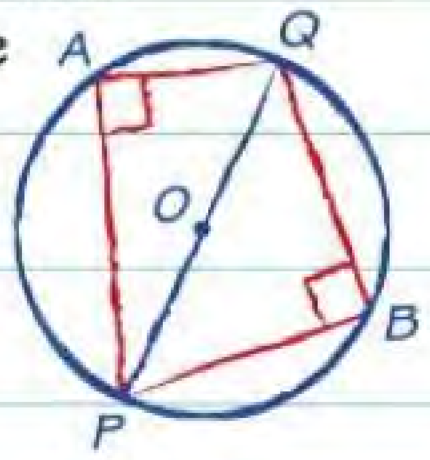
Had a look ☐ Nearly there ☐ Nailed it! ☐

Circle properties

You might need to solve circle problems involving **semicircles**, **tangents** and **chords**. Here are the three key facts you might need to use.

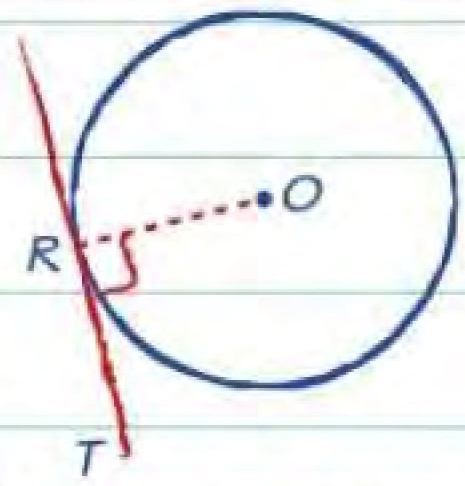
1 Angles in a semicircle

The angle in a semicircle is always a right angle.



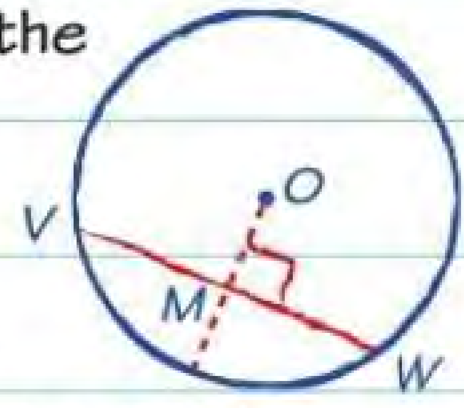
2 Chord and radius

The angle between a **tangent** and a **radius** is always a right angle.



3 Perpendicular to a chord

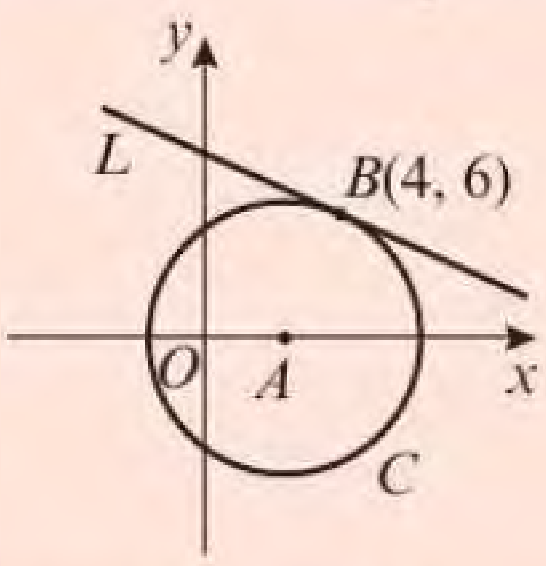
The **perpendicular** to a chord from the centre of the circle **bisects** that chord.



M is the midpoint of *VW*.

Worked example

The line L, with equation $x + 3y - 12 = 0$ is a tangent to the circle C with centre A. It touches the circle at the point $B(4, 6)$.



(a) Find an equation of the straight line through A and B. **(3 marks)**

$y = -\frac{1}{3}x + 4$ so L has gradient $-\frac{1}{3}$

So line through A and B has gradient 3.

$y - y_1 = m(x - x_1)$

$y - 6 = 3(x - 4)$

$y = 3x - 6$

(b) Given that A lies on the x-axis, find the coordinates of A. **(1 mark)**

At A, $y = 0$ so $0 = 3x - 6$ so $x = 2$

A has coordinates $(2, 0)$.

Worked example

The circle C has equation $(x - 2)^2 + (y - 1)^2 = 25$
The points $P(-1, 5)$ and $Q(5, -3)$ lie on the circle.

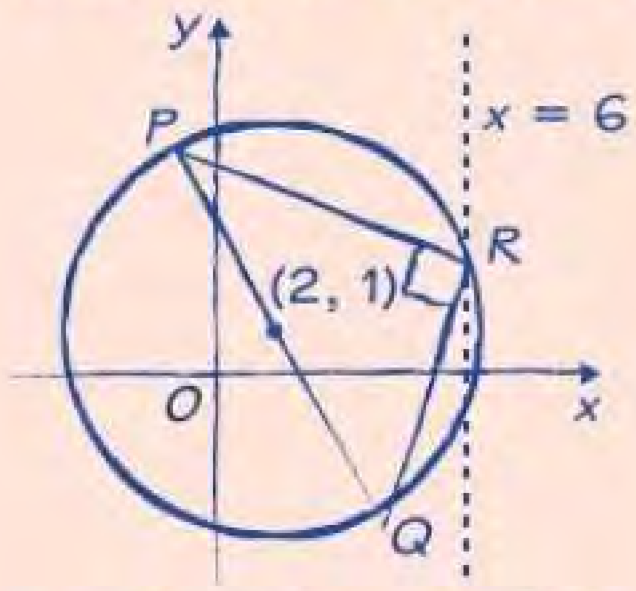
(a) Show that PQ is a diameter of C. **(2 marks)**

Midpoint of $PQ = \left(\left(\frac{-1 + 5}{2}\right), \left(\frac{5 + (-3)}{2}\right)\right)$
$= (2, 1)$

which is the centre of C, so PQ is a diameter.

(b) The point R has coordinates $(6, a)$, where $a > 0$ and the angle $PRQ = 90°$. Find the coordinates of R. **(4 marks)**

R lies on the circle because $\angle PRQ = 90°$.



It can help to draw a sketch.

$x = 6$ so $(6 - 2)^2 + (y - 1)^2 = 25$

$(y - 1)^2 = 9$

$y = -2$ or 4

$a > 0$ so R has coordinates $(6, 4)$.

Problem solved!

A tangent is perpendicular to a radius, so the straight line through A and B is perpendicular to L. Remember that if a line has gradient m, then any line perpendicular to it will have gradient $-\frac{1}{m}$.

Look at page 18 for a recap.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

The circle C has centre $(5, -2)$ and radius 10

(a) Write down an equation for C. **(2 marks)** Answer

(b) Verify that the point $(-1, 6)$ lies on C. **(1 mark)** Answer

(c) Find an equation of the tangent to C at the point $(-1, 6)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(4 marks)** Answer

Circles and lines

You can solve the equations of a circle and a straight line **simultaneously** to find the points of intersection. A straight line will intersect a circle once, twice or not at all. If a straight line just touches a circle then it is a **tangent** to the circle.

There is more about solving simultaneous equations on page 9.

Worked example

The circle C has centre the origin and radius 3. The straight line with equation $2x + y = k$, where k is a positive constant, is a tangent to C.

Find the exact value of k. **(6 marks)**

$y = k - 2x$ ①

$x^2 + y^2 = 9$ ②

Substitute ① into ②:

$x^2 + (k - 2x)^2 = 9$

$x^2 + 4x^2 - 4kx + k^2 = 9$

$5x^2 - 4kx + k^2 - 9 = 0$

Using the discriminant:

$b^2 - 4ac = 0$

$(-4k)^2 - 4(5)(k^2 - 9) = 0$

$16k^2 - 20k^2 + 180 = 0$

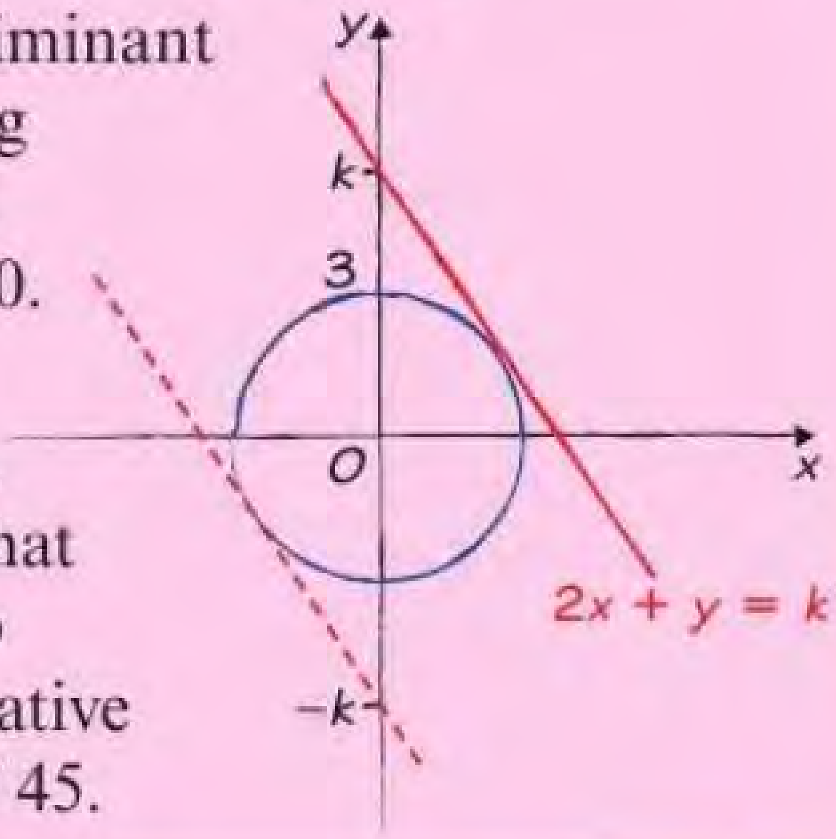
$k^2 = 45$

$k = 3\sqrt{5}$

Problem solved!

You can use the **discriminant** to solve this problem. You need to:

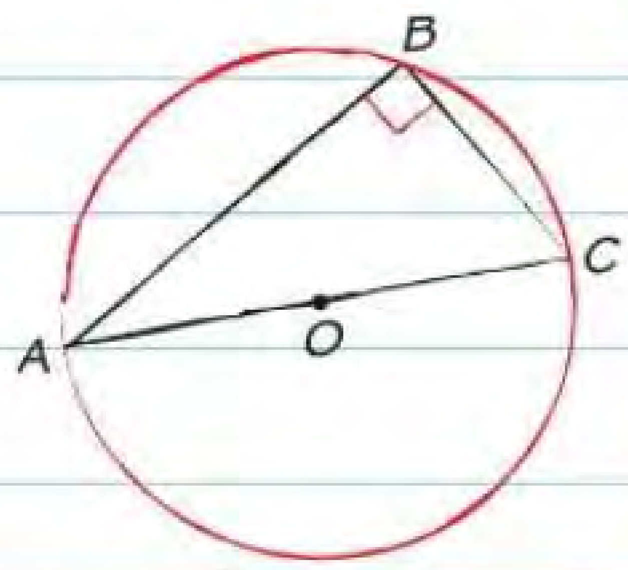
1. Write out the equation of the circle.
2. Substitute $y = k - 2x$ into this to solve both equations simultaneously.
3. Find the discriminant of the resulting quadratic and set it equal to 0.
4. Solve to find the value of k. You are told that k is positive so ignore the negative square root of 45.



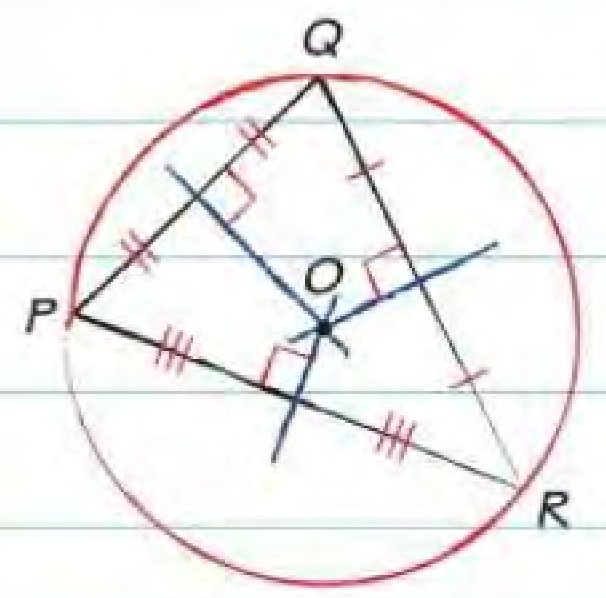
You will need to use problem-solving skills throughout your exam – **be prepared!**

Circumcircles

For any triangle, you can draw a unique circle which passes through all three vertices. This is called the circumcircle of the triangle



ABC is a right-angled triangle, so *AC* is a **diameter** of the circumcircle. Find the midpoint of the hypotenuse to find the centre of the circle.



For any triangle, *PQR*, the **perpendicular bisectors** of the sides will intersect at the centre of the circle.

There is more about properties of chords on page 21.

Now try this

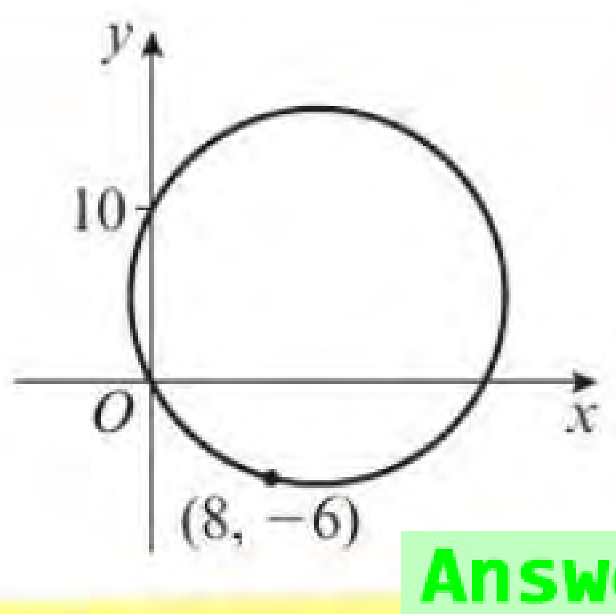
1 Find the coordinates of the points where the line with equation $y = x + 7$ intersects the circle with equation $x^2 + (y + 2)^2 = 45$.

Answer **(5 marks)**

2 The line with equation $2x - y + 2 = 0$ intersects the circle with centre $(k, 0)$ and radius 2 at two distinct points. Find the range of possible values of k, giving your answer in surd form.

Answer **(7 marks)**

3 A circle passes through the points (0, 0), (0, 10) and (8, –6), as shown in the diagram. Find an equation for the circle. **(7 marks)**



Answer

The perpendicular bisectors of two chords will intersect at the centre of the circle.