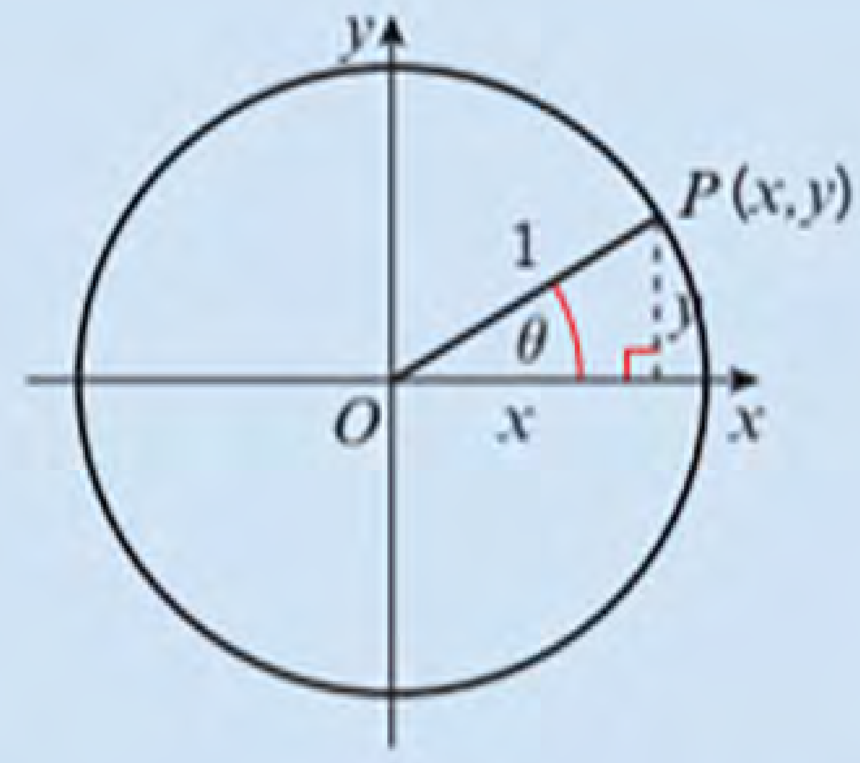
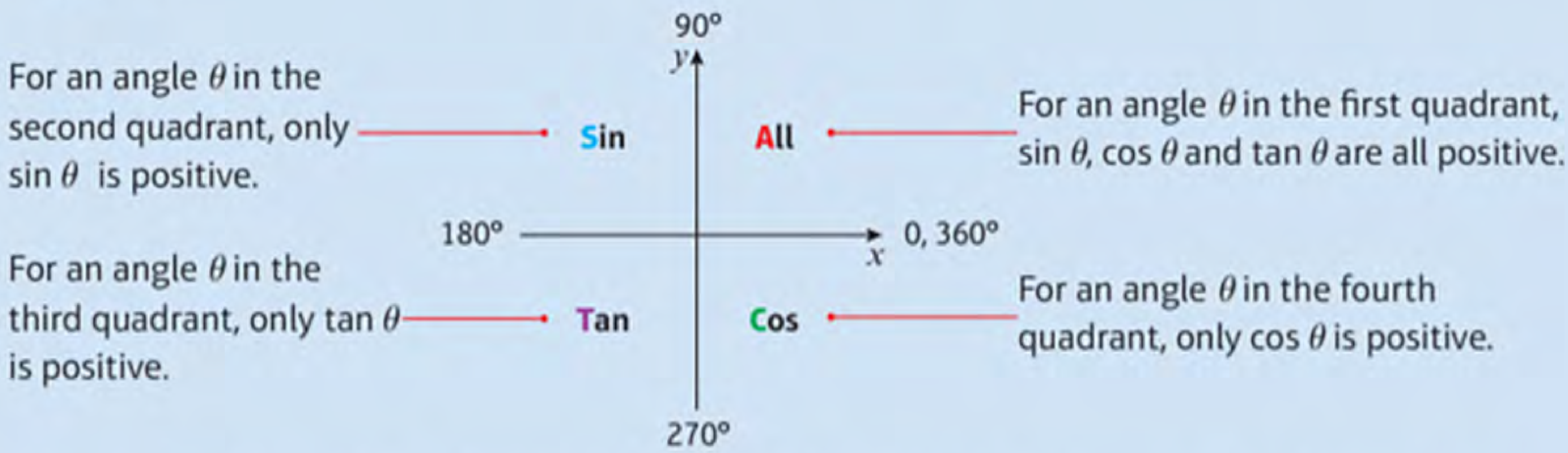
Summary of key points

1 For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:

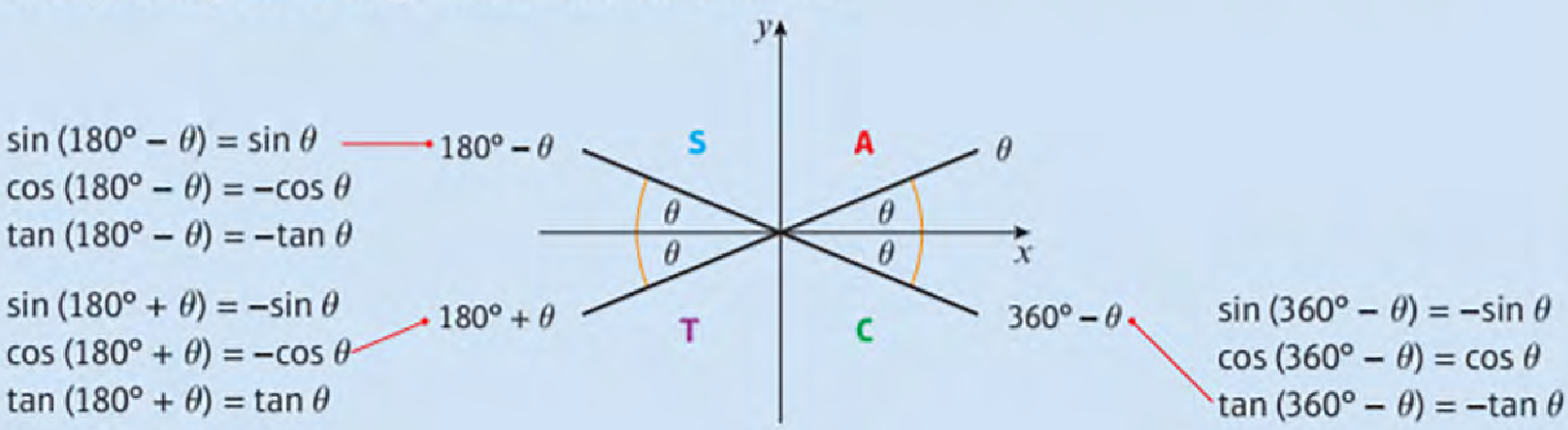
- $\cos\theta = x = x$-coordinate of P
- $\sin\theta = y = y$-coordinate of P
- $\tan\theta = \frac{y}{x} =$ gradient of OP



2 You can use the quadrants to determine whether each of the trigonometric ratios is positive or negative.



3 You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding **acute** angle made with the x-axis, θ.



4 The trigonometric ratios of 30°, 45° and 60° have exact forms, given below:

$\sin 30° = \frac{1}{2}$ $\quad\cos 30° = \frac{\sqrt{3}}{2}$ $\quad\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad\tan 45° = 1$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\quad\cos 60° = \frac{1}{2}$ $\quad\tan 60° = \sqrt{3}$

5 For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$

6 For all values of θ such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

7
- Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leqslant k \leqslant 1$
- Solutions to $\tan\theta = p$ exist for all values of p.

8 When you use the inverse trigonometric functions on your calculator, the angle you get is called the **principal value**.

9 Your calculator will give principal values in the following ranges:
- $\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$
- $\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$
- $\tan^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$

Had a look ☐ Nearly there ☐ Nailed it! ☐

Trigonometric identities

You might need to use one of these two trigonometric identities to **simplify** a trig equation before solving it. They are true for **all values** of x or θ.

1. $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$
2. $\sin^2\theta + \cos^2\theta \equiv 1$

$\sin^2\theta$ means $(\sin\theta)^2$.

Quadratic equations

If an equation involves $\sin^2\theta$ and $\sin\theta$ (or $\cos^2\theta$ and $\cos\theta$) then it is a quadratic. You can solve it by factorising:

$$2\sin^2\theta - 3\sin\theta - 2 = 0$$
$$(2\sin\theta + 1)(\sin\theta - 2) = 0$$

$\sin\theta = -\frac{1}{2}$ $\quad$ $\sin\theta = 2$

No solutions exist to $\sin\theta = 2$, so you would only solve $\sin\theta = -\frac{1}{2}$.

Golden rules

When finding solutions to **quadratic trigonometric** equations, remember these golden rules:

1. Write everything in terms of $\sin^2\theta$ and $\sin\theta$ (or $\cos^2\theta$ and $\cos\theta$).
2. Solutions to $\sin x = k$ and $\cos x = k$ **only exist** if $-1 \leqslant k \leqslant 1$. Solutions to $\tan x = k$ exist for **any value** of k.

Worked example

(a) Show that the equation $5\cos x = 1 + 2\sin^2 x$ can be written in the form $2\cos^2 x + 5\cos x - 3 = 0$ **(2 marks)**

$5\cos x = 1 + 2\sin^2 x$
$= 1 + 2(1 - \cos^2 x)$
$= 1 + 2 - 2\cos^2 x$
$2\cos^2 x + 5\cos x - 3 = 0$

(b) Solve this equation for $0 \leqslant x < 360°$. **(3 marks)**

$(2\cos x - 1)(\cos x + 3) = 0$

$\cos x = \frac{1}{2}$ $\quad$ ~~$\cos x = -3$~~

$\cos^{-1}\left(\frac{1}{2}\right) = 60°$

$-60° + 360° = 300°$

$x = 60°, 300°$

You can use $\sin^2 x + \cos^2 x = 1$ to rewrite $\sin^2 x$ in terms of $\cos^2 x$:

$\sin^2 x = 1 - \cos^2 x$

You can then rearrange to get a quadratic equation in $\cos x$. You might find it easier to factorise if you write it as:

$2C^2 + 5C - 3 = 0 \rightarrow (2C - 1)(C + 3) = 0$

If $\cos x + 3 = 0$ then $\cos x = -3$, which has **no solutions**, so you can ignore the second factor.

Worked example

Given that $\sin\theta = 4\cos\theta$, find the value of $\tan\theta$. **(1 mark)**

$\dfrac{\sin\theta}{\cos\theta} = 4$ so $\tan\theta = 4$

Now try this

1 Find all the solutions, in the interval $0 \leqslant x < 360°$, of the equation

$$3\cos^2 x - 9 = 11\sin x$$

giving each solution correct to 1 decimal place. **(6 marks)** Answer

Use $\cos^2 x = 1 - \sin^2 x$ to get a quadratic equation in $\sin x$.

2 (a) Show that the equation $5\sin x = 2\tan x$ can be written in the form

$$\sin x(5\cos x - 2) = 0$$ **(2 marks)** Answer

(b) Solve, for $0 \leqslant x < 360°$, $5\sin x = 2\tan x$ **(4 marks)** Answer

Start by writing $\tan x$ as $\dfrac{\sin x}{\cos x}$

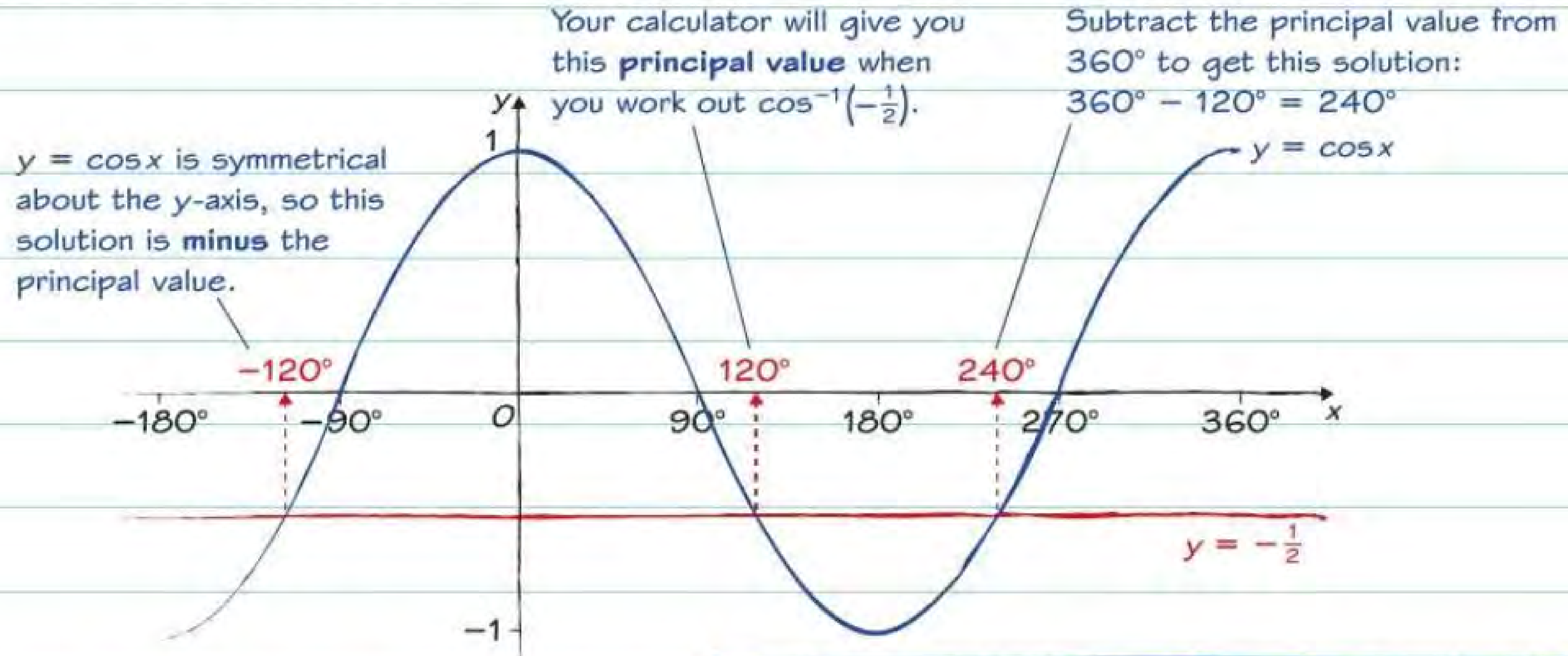
Either $\sin x = 0$, or $5\cos x - 2 = 0$. Both these factors will give you solutions.

Trigonometric equations 1

You can solve an equation involving **sin**, **cos** or **tan**. You need to be really careful because these equations can have **multiple solutions**. You will be given a **range** (or **interval**) of values for x. You need to find values of x that are in that range.

Using graphs to find solutions

This graph shows the solutions to the equation $\cos x = -\frac{1}{2}$ in the range $-180° \leqslant x \leqslant 360°$.



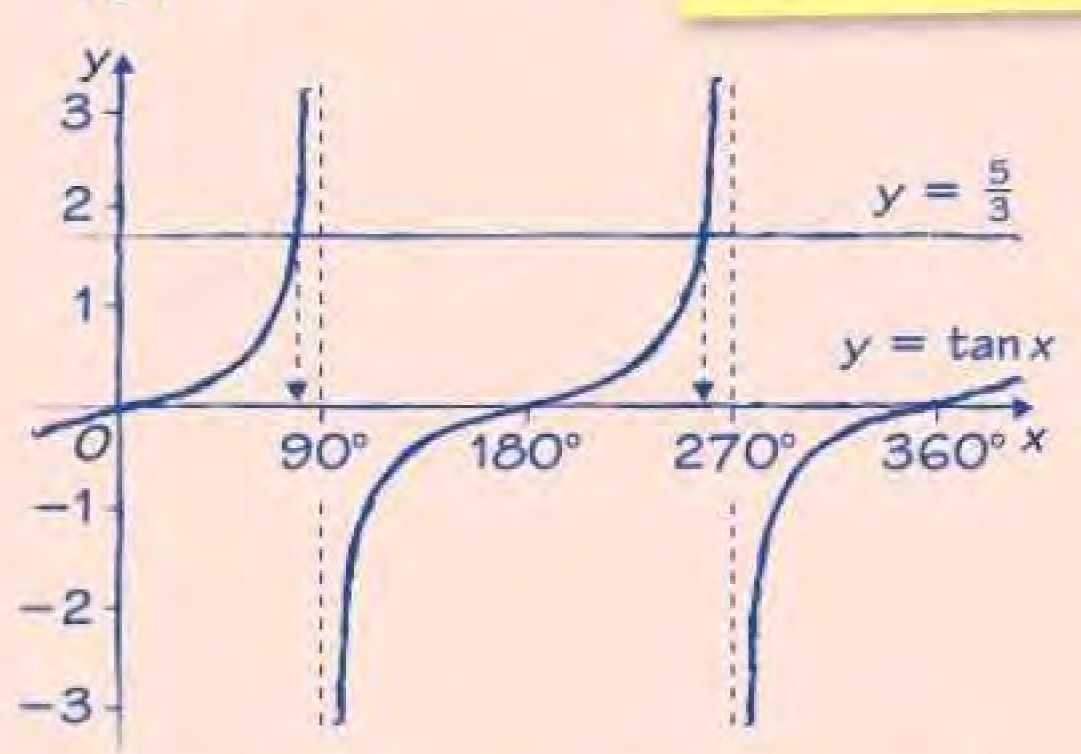
Worked example

Solve $3\tan x = 5$ in the interval $0 \leqslant x < 360°$.
Give your answers to 1 decimal place.
(3 marks)

$\tan x = \frac{5}{3}$

$\tan^{-1}\left(\frac{5}{3}\right) = 59.036...°$

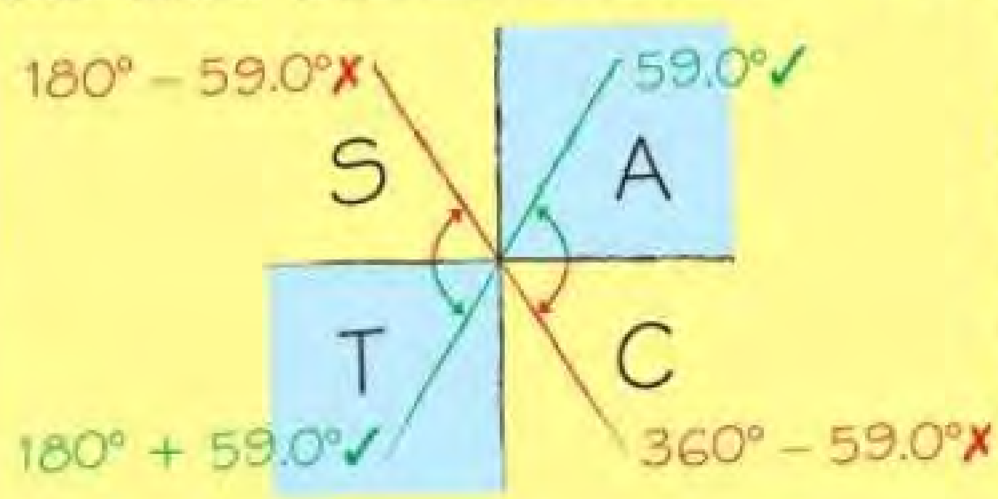
Work to 3 d.p. then round your final answer.



$180° + 59.036...° = 239.036...°$

$x = 59.0°, 239.0°$ (1 d.p.)

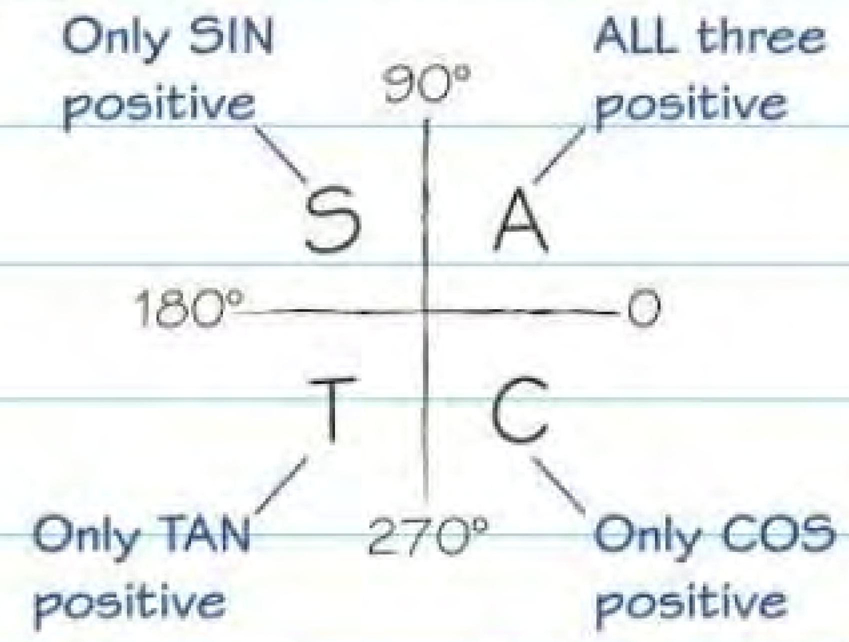
Use your calculator to find the **principal value** of x. You can find the other solution by sketching a graph or by drawing straight lines like this on a CAST diagram. You know that tan x is **positive**, so the other solution must be in the **third quadrant**.



Tan x is **only** positive for angles in the first and third quadrants. So you can **reject** the angles in the second and fourth quadrants.

Using a CAST diagram

A **CAST** diagram tells you which trigonometric ratios are **positive** in which **quadrant**.



Now try this

1 (a) Sketch the graph of $y = \sin x$ in the interval $0 \leqslant x < 360°$. **(2 marks)**

(b) Find the values of x in the interval $0 \leqslant x \leqslant 360°$ for which $\sin x = -0.3$ Give your answers correct to 1 decimal place. **(3 marks)**

Answer

Answer

2 Solve, for $-180° \leqslant \theta < 180°$, the equation

(a) $3\cos\theta = 1$ **(3 marks)** Answer

(b) $\tan\theta + 2 = 0$ **(3 marks)** Answer

There will be **two solutions** to each equation. Find one using your calculator, then sketch the graph to find the other.

Trigonometric equations 2

You need to be careful if a trigonometric equation involves a **function** of x or θ.

Worked example

Find the exact solutions of the equation

$\cos(\theta - 50°) = 0.5$

in the interval $0 \leqslant \theta < 180°$. **(3 marks)**

$0 \leqslant \theta < 180°$ so $-50° \leqslant \theta - 50° < 130°$

Let $Z = \theta - 50°$

$\cos Z = 0.5$, $-50° \leqslant Z < 130°$

So $Z = \cos^{-1}(0.5) = 60°$ in range ✓

or $Z = -60°$ not in range ✗

or $Z = 360° - 60° = 300°$ not in range ✗

So $\theta - 50° = 60°$

So $\theta = 110°$

Transforming the range

The safest way to solve an equation involving sin, cos or tan of $(x + b)$ or (ax) is to **transform the range**.

$0 \leqslant \theta < 180°$

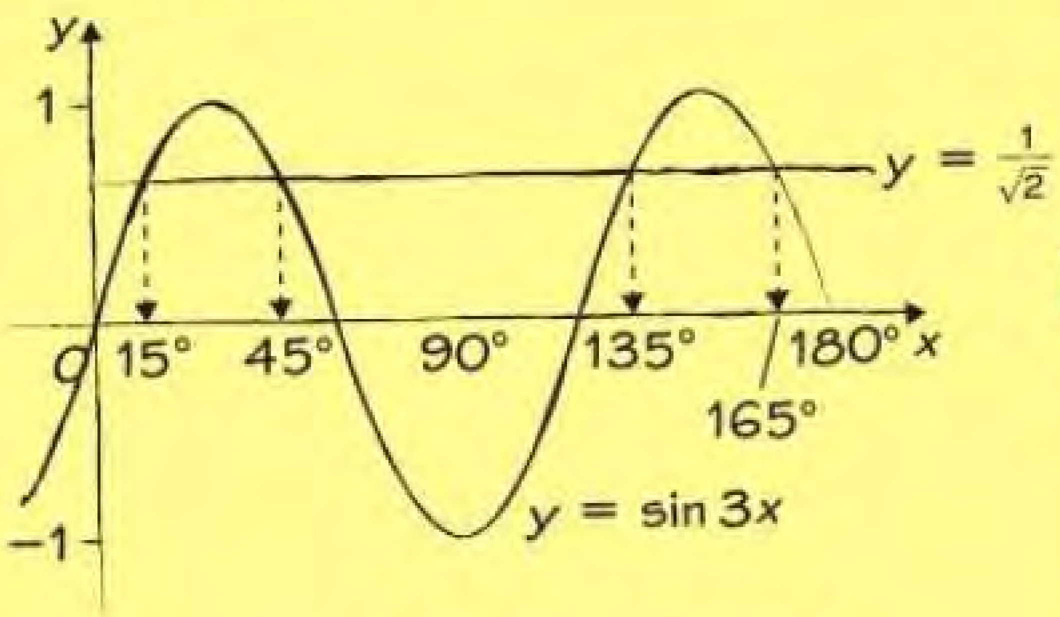
$-50°$ $-50°$

$-50° \leqslant \theta - 50° < 130°$

If $Z = \theta - 50°$, you need to find all the values of Z such that $\cos Z = 0.5$ in the range $-50° \leqslant Z < 130°$. You can then find the corresponding value of θ for each solution.

The graph of $y = \sin(3x)$ is a horizontal stretch of the graph of $y = \sin x$ with scale factor $\frac{1}{3}$.

This sketch shows the solutions of the equation.



Make sure you remember to transform your solutions for $3x$ back into solutions for x at the end, and double check that they all lie within $0 \leqslant x < 180°$.

Worked example

Solve, for $0 \leqslant x < 180°$, the equation $\sin(3x) = \frac{1}{\sqrt{2}}$ **(6 marks)**

$0 \leqslant x < 180°$ so $0 \leqslant 3x < 540°$

Let $Z = 3x$

$\sin Z = \frac{1}{\sqrt{2}}$, $0 \leqslant Z < 540°$

So $Z = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$ in range ✓

or $Z = 45° + 360° = 405°$ in range ✓

or $Z = 45° + 720° = 765°$ not in range ✗

or $Z = 180° - 45° = 135°$ in range ✓

or $Z = 180° - 45° + 360° = 495°$ in range ✓

or $Z = 180° - 45° + 720° = 855°$ not in range ✗

So $3x = 45°$ or $405°$ or $135°$ or $495°$

So $x = 15°$ or $135°$ or $45°$ or $165°$

Now try this

1 Solve, for $0 \leqslant x < 360°$

(a) $\sin(x - 40°) = -\frac{1}{2}$ Answer **(4 marks)**

(b) $\cos(2x) = \frac{\sqrt{3}}{2}$ Answer **(4 marks)**

2 Find all the solutions of the equation $\cos^2(x + 30°) = \frac{1}{4}$ in the range $-180° \leqslant x \leqslant 180°$. Answer **(6 marks)**

You need to consider the positive and negative square roots separately. This is like solving two separate equations:

$\cos(x + 30°) = \frac{1}{2}$ and

$\cos(x + 30°) = -\frac{1}{2}$