Summary of key points

1 For all real values of x:

- If $f(x) = e^x$ then $f'(x) = e^x$
- If $y = e^x$ then $\frac{dy}{dx} = e^x$

2 For all real values of x and for any constant k:

- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

3 $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$

4 **The laws of logarithms:**

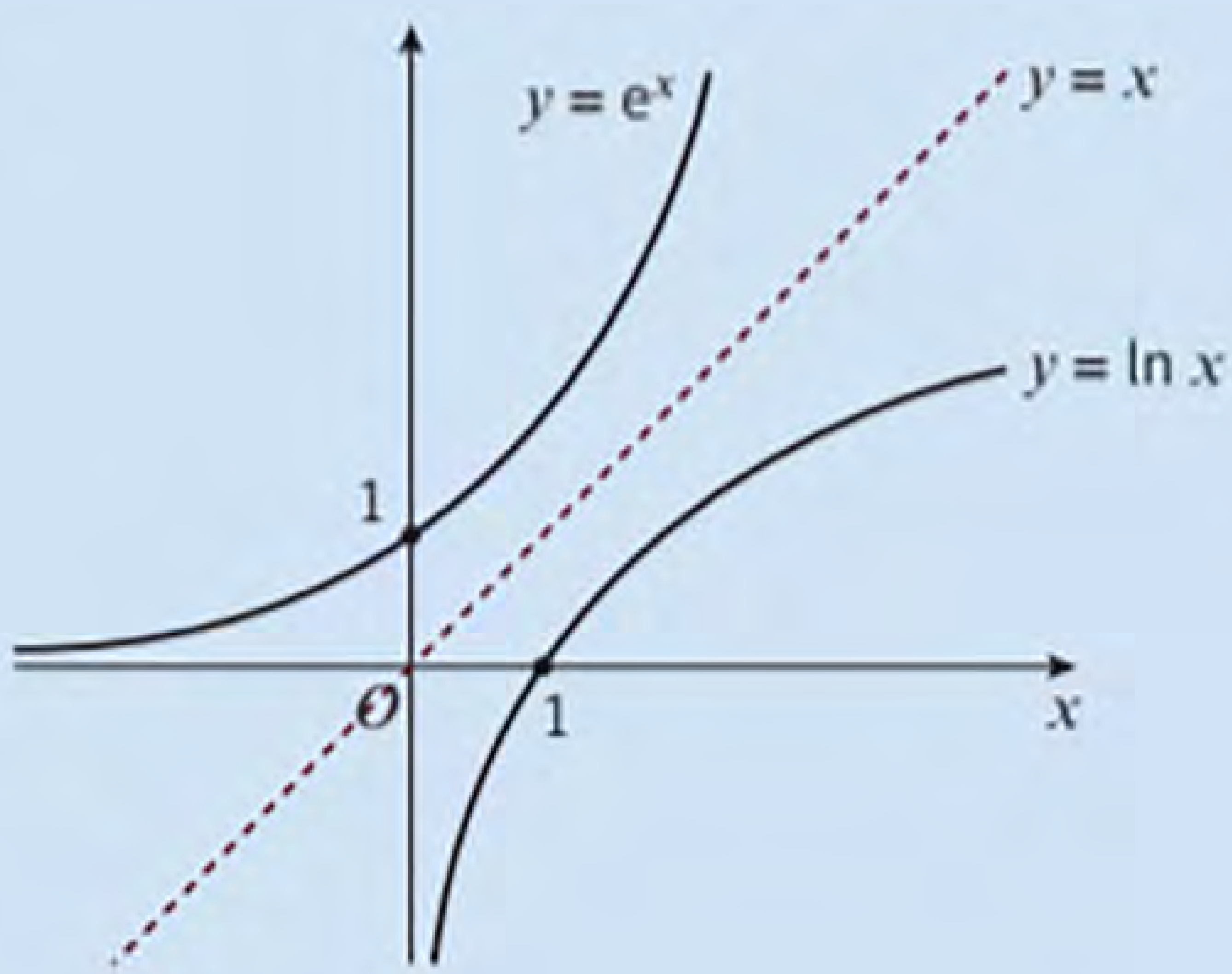
- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ (the division law)
- $\log_a (x^k) = k \log_a x$ (the power law)

5 You should also learn to recognise the following special cases:

- $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $(a > 0, a \neq 1)$
- $\log_a 1 = 0$ $(a > 0, a \neq 1)$

6 Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$

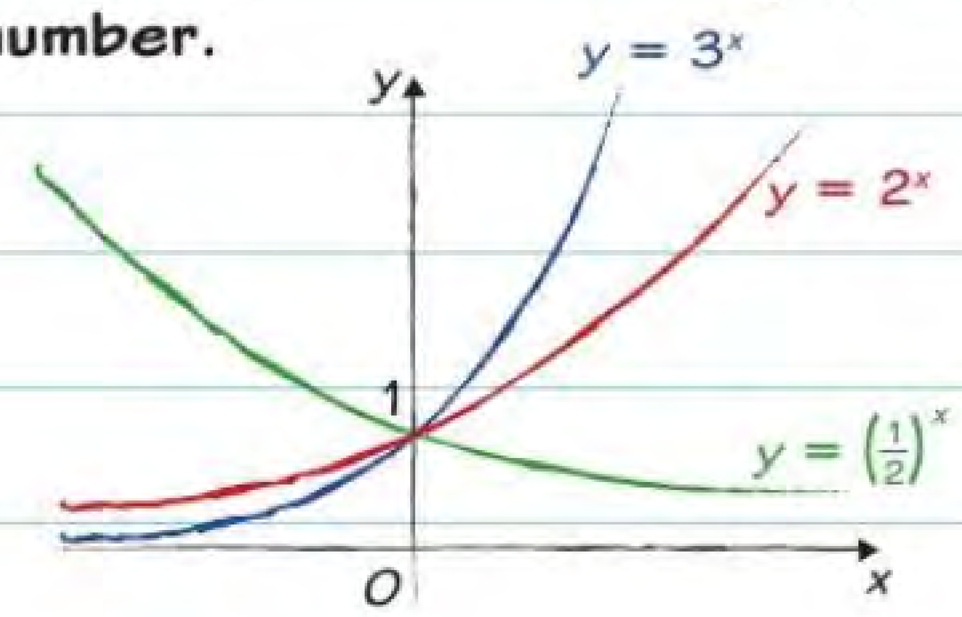
7 The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.



8 $e^{\ln x} = \ln (e^x) = x$

Exponential functions

You need to be able to sketch the graph of $y = a^x$. You can only sketch this graph when a is a **positive number**.



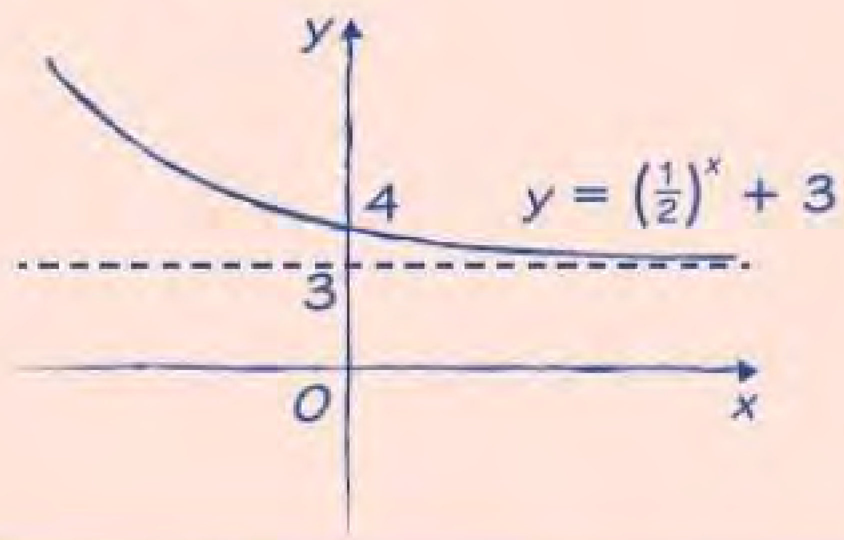
$y = a^x$

- ✓ Passes through (0, 1).
- ✓ $y = 0$ is an **asymptote**.
- ✓ If $a > 1$ graph curves **upwards**.
- ✓ If $0 < a < 1$ graph curves **downwards**.

Worked example

Sketch the graph of

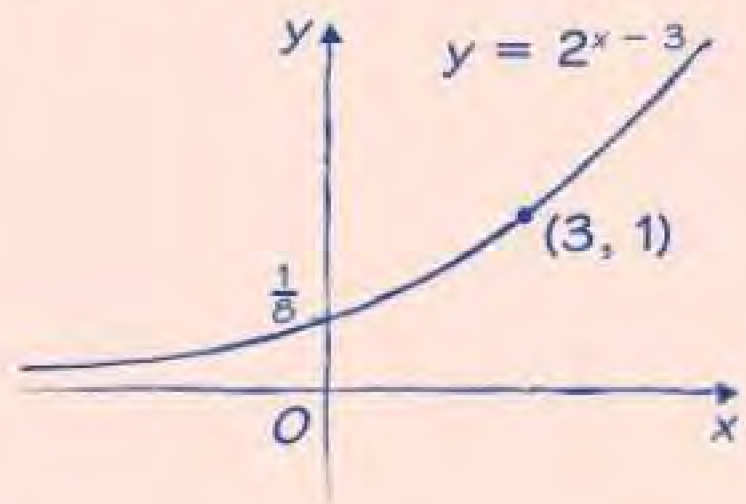
(a) $y = \left(\frac{1}{2}\right)^x + 3$ **(3 marks)**



This is a translation of the graph $y = \left(\frac{1}{2}\right)^x$ by the vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$. The asymptote at $y = 0$ is translated to $y = 3$.

There is more about transformations of graphs on pages 13 and 14.

(b) $y = 2^{x-3}$ **(3 marks)**



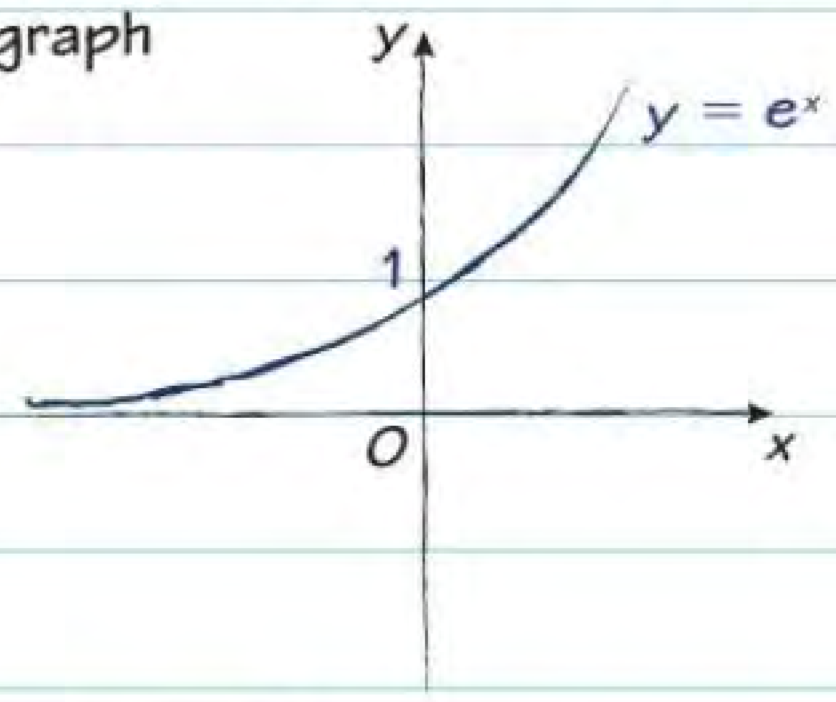
This is a translation of the graph $y = 2^x$ by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$. You should label any points where the curve cuts the axes.

When $x = 0$, $y = 2^{-3} = \frac{1}{8}$

$y = e^x$

The letter 'e' represents a constant number. It is approximately equal to 2.718 28...

This is the graph of $y = e^x$



Golden rule

If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

Worked example

Differentiate with respect to x

(i) e^x (ii) e^{3x} (iii) $5e^{\frac{x}{2}}$ **(3 marks)**

$y = e^x$, $\frac{dy}{dx} = e^x$

$y = e^{3x}$, $\frac{dy}{dx} = 3e^{3x}$

$y = 5e^{\frac{1}{2}x}$, $\frac{dy}{dx} = \frac{5}{2}e^{\frac{1}{2}x}$

Now try this

1 Match each of these equations to one of the graphs on the right:

(a) $y = 0.2^x - 2$ (b) $y = -(5^x)$

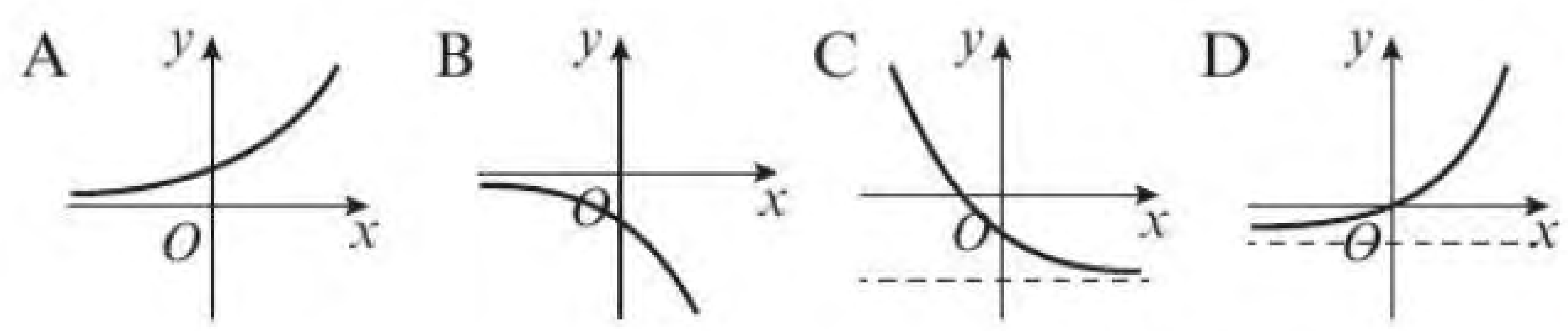
(c) $y = 2^x - 1$ (d) $y = \frac{1}{3}^{1-x}$

Answer

Answer

Answer

Answer



2 Differentiate with respect to x

(a) e^{-2x} **(1 mark)** (b) $2e^{5x}$ **(1 mark)** (c) $e^{\frac{x}{3}}$ **(1 mark)**

Answer Answer Answer

Logarithms

Logarithms (or **logs**) are a way of writing facts about **powers**. These two statements mean the same thing:

You say 'log to the base *a* of *b* equals *x*'.

$$\log_a b = x \longleftrightarrow a^x = b$$

a is the **base** of the logarithm.

For example: $\log_3 9 = 2 \longleftrightarrow 3^2 = 9$

Remembering the order

The key to being confident in log questions is remembering the **basic definition**.

Start at the **base**, and work in a **circle**.

$$\log_a b = x$$

Raise *a* to the power *x*... ... to get *b*

Laws of logarithms

Learn these four key laws for manipulating expressions involving logs. These laws all work for logarithms with **the same base**.

1. $\log_a x + \log_a y = \log_a (xy)$

   $\log_4 8 + \log_4 2 = \log_4 16 = 2$ (since $4^2 = 16$)

2. $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

   $\log_9 18 - \log_9 6 = \log_9 3 = \frac{1}{2}$ (since $9^{\frac{1}{2}} = 3$)

3. $\log_a \left(\frac{1}{x}\right) = -\log_a x$

   $\log_8 \left(\frac{1}{2}\right) = -\log_8 2 = -\frac{1}{3}$

4. $\log_a (x^n) = n\log_a x$

   $\log_5 (25^3) = 3\log_5 25 = 3 \times 2 = 6$

Worked example

Find

(a) the positive value of *x* such that $\log_x 49 = 2$ **(2 marks)**

$x^2 = 49$

$x = 7$

(b) the value of *y* such that $\log_5 y = -2$ **(2 marks)**

$5^{-2} = y$

$y = \frac{1}{25}$

Write down the corresponding power fact. Remember: $\log_a b = x \longleftrightarrow a^x = b$

Worked example

Express $3\log_a 2 + \log_a 10$ as a single logarithm to base *a*. **(3 marks)**

$$3\log_a 2 + \log_a 10 = \log_a (2^3) + \log_a 10$$
$$= \log_a (2^3 \times 10)$$
$$= \log_a 80$$

Use law 4 to write $3\log_a 2$ as $\log_a (2^3)$, then use law 1 to combine the two logarithms.

Special cases

Learn these two special cases to save time:

1. $\log_a a = 1$
2. $\log_a 1 = 0$

Now try this

1 Find

(a) the value of *y* such that $\log_3 y = -1$ **(2 marks)**

(b) the value of *p* such that $\log_p 8 = 3$ **(2 marks)**

(c) the value of $\log_4 8$ **(2 marks)**

2 Express as a single logarithm to base *a*

(a) $2\log_a 5$ **(2 marks)**

(b) $\log_a 2 + \log_a 9$ **(2 marks)**

(c) $3\log_a 4 - \log_a 8$ **(3 marks)**

3 Show that $2\log_8 6 - \log_8 9 = \frac{2}{3}$

Equations with logs

If you see an equation involving **logarithms** in your exam, you will probably need to rearrange it using the **laws of logarithms**, which are covered on page 48.

Two steps to solving log equations

Follow these two steps to solve most log equations in your exam:

1. Group the log terms on one side, then use the laws of logs on page 48 to write them as a **single algorithm.**
2. Rewrite $\log_a f(x) = k$ as $f(x) = a^k$ and solve the equation to find x.

Undefined logs

The value $\log_a b$ is **only defined** for $b > 0$. You can't calculate $\log_a 0$ or the log of any negative number. If an equation contains $\log_a x$ or $\log_a kx$ then **ignore** any solutions where $x \leqslant 0$.

If there are solutions to ignore in an exam question, you will usually be given a **range** of possible values for x.

Worked example

Solve the equation $2\log_5 x - \log_5 3x = 2$ **(4 marks)**

$\log_5 x^2 - \log_5 3x = 2$

$\log_5\left(\frac{x^2}{3x}\right) = 2$

$\frac{x^2}{3x} = 5^2 = 25$

$x^2 = 75x$

$x^2 - 75x = 0$

$x(x - 75) = 0$ so ~~$x = 0$~~ or $\underline{x = 75}$

Follow the two steps given above:

1. Rearrange the left-hand side into a single logarithm.
2. Write the corresponding power fact: $\log_5 f(x) = 2 \rightarrow f(x) = 5^2$

Worked example

Given that $0 < x < 2$ and $\log_3(2 - x) - 2\log_3 x = 1$, find the value of x. **(6 marks)**

$\log_3(2 - x) - \log_3(x^2) = 1$

$\log_3\left(\frac{2 - x}{x^2}\right) = 1$

$\frac{2 - x}{x^2} = 3$

$2 - x = 3x^2$

$3x^2 + x - 2 = 0$

$(3x - 2)(x + 1) = 0$

$\underline{x = \frac{2}{3}}$ ~~$x = -1$~~

Ignore this solution because $0 < x < 2$.

You need to solve two simultaneous **equations**. You can ignore the negative square root because p and q are positive.

Worked example

p and q are positive constants, with

$p = 5q$ ①

$\log_5 p + \log_5 q = 2$ ②

Find the exact values of p and q. **(6 marks)**

From ②: $\log_5(pq) = 2$

Substituting ①: $\log_5(5q^2) = 2$

$5q^2 = 5^2 = 25$

$q^2 = 5$

$q = \sqrt{5}$

Substituting into ①: $p = 5\sqrt{5}$

Now try this

Answer

1 Solve $\log_2(x + 1) - \log_2 x = \log_2 5$ **(3 marks)**

2 Solve the equation $\log_6(x - 1) + \log_6 x = 1$ **(4 marks)**

Answer

Answer

3 Solve $\log_3(x - 1) = -1$ **(2 marks)**

4 Find the values of x such that $2\log_4 x - \log_4(x - 3) = 2$ **(5 marks)**

Answer

Exponential equations

You can find unknown powers in equations using the log functions on your **calculator**. Make sure you **write down** any logarithms you are working out.

[$\log_\blacksquare \square$] You can use this key to work out logs to any base.

[log] This key means $\log_{10}$. You can **take logs** of both sides of an equation and solve it using this key.

Worked example

(a) Solve the equation $4^x = 13$, giving your answer to 3 significant figures. **(3 marks)**

$x = \log_4 13 = 1.85$ (3 s.f.)

(b) Find, to 3 significant figures, the value of y for which $5^y = 4$ **(3 marks)**

$\log(5^y) = \log 4$

$y\log 5 = \log 4$

$y = \dfrac{\log 4}{\log 5} = 0.861$ (3 s.f.)

Make sure you **write down** the logarithm you need to find even if you are working it out on your calculator in one go. Part (a) shows a method using the [$\log_\blacksquare \square$] key. Part (b) shows a method by **taking logs** of both sides and using the laws of logs. This works for any base, so you can use the [log] key on your calculator.

Worked example

Solve the equation

$5^{2x} - 3(5^x) + 2 = 0$

giving your answers to 2 decimal places where appropriate. **(6 marks)**

$(5^x)^2 - 3(5^x) + 2 = 0$

$(5^x - 2)(5^x - 1) = 0$

$5^x = 2$ $\quad$ $5^x = 1$

$x = \log_5 2 = 0.43$ (2 d.p.) $\quad$ $x = \log_5 1 = 0$

Use the fact that $5^{2x} = (5^x)^2$ to write a **quadratic equation** in 5^x. For a reminder on the **laws of indices** have a look at page 1. It might help to write the equation as $Y^2 - 3Y + 2 = 0$, with $Y = 5^x$.

Factorising gives you two values for 5^x. Each of these gives you a value for x. Remember that $\log_a 1 = 0$ for any base, so $\log_5 1 = 0$.

Worked example

Solve the equation $2^x = 5^{x-3}$, giving your answer to 3 significant figures. **(4 marks)**

$\log 2^x = \log 5^{x-3}$

$x\log 2 = (x - 3)\log 5$

$3\log 5 = x\log 5 - x\log 2$

$3\log 5 = x(\log 5 - \log 2)$

$x = \dfrac{3\log 5}{\log 5 - \log 2} = 5.27$ (3 s.f.)

Problem solved!

This exponential equation has different bases on each side. You can still take logs of both sides but you must take logs **to the same base** on each side.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

1 Find, to 3 significant figures

(a) the value of b for which $2^b = 15$ **(3 marks)** Answer

(b) the value of x for which $6^x = 0.4$ **(3 marks)** Answer

2 (a) Solve the equation $3^{2x} + 3^x = 6$, giving your answer to 2 decimal places. **(6 marks)** Answer

(b) Explain why there is only one solution to the equation $3^{2x} + 3^x = 6$ **(1 mark)** Answer

3 Solve $3^{x-1} = 2^{2x+1}$, giving your answer correct to 3 significant figures. **(4 marks)** Answer

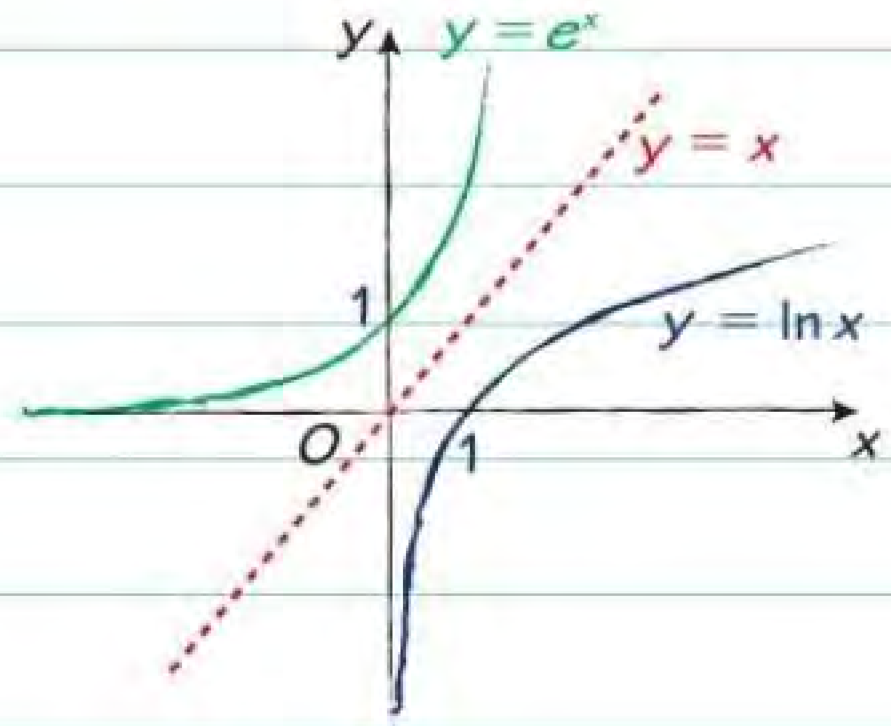
4 $f(x) = 6^{x^2}, x \in \mathbb{R}$ $\quad$ $g(x) = 2^{x-1}, x \in \mathbb{R}$

Show that the curves $y = f(x)$ and $y = g(x)$ do not intersect. **(5 marks)** Answer

Set $6^{x^2} = 2^{x-1}$ and use the discriminant of the resulting quadratic equation in x.

Natural logarithms

A logarithm to the base e is sometimes called a natural logarithm. The graph of $y = \ln x$ is the graph of $y = e^x$ reflected in the line $y = x$. It has an asymptote at $x = 0$ and crosses the x-axis at (1, 0).



Golden rule

$\ln x$ works just like any other logarithm: $e^{\ln x} = \ln(e^x) = x$

Look for the $e^{\blacksquare}$ and ln functions on your calculator to find values of e^x and $\ln x$.

Worked example

Find the exact solutions to the equation

$e^x + 3e^{-x} = 4$ **(4 marks)**

$(e^x)^2 + 3 = 4e^x$

$(e^x)^2 - 4e^x + 3 = 0$

$(e^x - 3)(e^x - 1) = 0$

$e^x = 3$, $x = \ln 3$

$e^x = 1$, $x = \ln 1 = 0$

You can write this as a quadratic equation using the substitution $u = e^x$:

$u + \frac{3}{u} = 4$

$u^2 - 4u + 3 = 0$

If a question asks for exact solutions then you should leave your answers as logs, or powers of e, and simplify them as much as possible.

Problem solved!

The laws of logarithms work exactly the same with ln as they do with $\log_{10}$ and $\log_a$

You can't combine the $2x$ with the e^{3x+1} easily, so just take logs of both sides. You can then use the laws of logs to simplify the left-hand side. Group the x terms together, then factorise to get x on its own.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

Solve $2^x e^{3x+1} = 10$, giving your answer in the form $\frac{a + \ln b}{c + \ln d}$ where a, b, c and d are integers to be found. **(5 marks)**

$\ln(2^x e^{3x+1}) = \ln 10$

$\ln 2^x + \ln(e^{3x+1}) = \ln 10$

$x \ln 2 + 3x + 1 = \ln 10$

$x(\ln 2 + 3) = \ln 10 - 1$

$x = \frac{-1 + \ln 10}{3 + \ln 2}$

Now try this

1 Given that $f(x) = \ln x$, $x > 0$, sketch, on separate axes, the graphs of

(a) $y = f(x - 2)$ Answer **(2 marks)**

(b) $y = -f(x)$ Answer **(2 marks)**

(c) $y = f(3x)$ Answer **(2 marks)**

2 The point P with y-coordinate 6 lies on the curve with equation $y = 3e^{2x-1}$. Find, in terms of ln 2, the x-coordinate of P. **(2 marks)** Answer

3 Solve

(a) $\ln(x + 1) - \ln x = \ln 5$ Answer **(2 marks)**

(b) $e^{4x} + 3e^{2x} = 10$ Answer **(5 marks)**

(c) $\ln(6x + 7) = 2\ln x$, $x > 0$ **(4 marks)** Answer

4 Find the exact solution to the equation

$3^x e^{2x-5} = 7$ **(5 marks)** Answer

5 The function f is defined by

$f : x \mapsto \frac{3x^2 - 7x + 2}{x^2 - 4}$, $x \neq \pm 2$

(a) Show that $f(x) = \frac{3x - 1}{x + 2}$ **(3 marks)** Answer

(b) Hence, or otherwise, solve the equation $\ln(3x^2 - 7x + 2) = 1 + \ln(x^2 - 4)$, $x > 2$ giving your answer in terms of e. **(4 marks)** Answer

Start with $6 = 3e^{2x-1}$. Divide both sides by 3 then take the natural logarithm of both sides.