Summary of key points

1 The **gradient** of a **curve** at a given point is defined as the gradient of the **tangent** to the curve at that point.

2 The **gradient function**, or **derivative**, of the curve $y = \mathrm{f}(x)$ is written as $\mathrm{f}'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

3 For all real values of n, and for a constant a:

- If $\mathrm{f}(x) = x^n$ then $\mathrm{f}'(x) = nx^{n-1}$
- If $\mathrm{f}(x) = ax^n$ then $\mathrm{f}'(x) = anx^{n-1}$
- If $y = x^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$
- If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$

4 For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$$

5 If $y = \mathrm{f}(x) \pm \mathrm{g}(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) \pm \mathrm{g}'(x)$.

6 The tangent to the curve $y = \mathrm{f}(x)$ at the point with coordinates $(a, \mathrm{f}(a))$ has equation

$$y - \mathrm{f}(a) = \mathrm{f}'(a)(x - a)$$

7 The normal to the curve $y = \mathrm{f}(x)$ at the point with coordinates $(a, \mathrm{f}(a))$ has equation

$$y - \mathrm{f}(a) = -\frac{1}{\mathrm{f}'(a)}(x - a)$$

8
- The function $\mathrm{f}(x)$ is **increasing** on the interval $[a, b]$ if $\mathrm{f}'(x) \geqslant 0$ for all values of x such that $a < x < b$.
- The function $\mathrm{f}(x)$ is **decreasing** on the interval $[a, b]$ if $\mathrm{f}'(x) \leqslant 0$ for all values of x such that $a < x < b$.

9 Differentiating a function $y = \mathrm{f}(x)$ twice gives you the second order derivative, $\mathrm{f}''(x)$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

10 Any point on the curve $y = \mathrm{f}(x)$ where $\mathrm{f}'(x) = 0$ is called a **stationary point**. For a small positive value h:

Type of stationary point	$\mathrm{f}'(x-h)$	$\mathrm{f}'(x)$	$\mathrm{f}'(x+h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

11 If a function $\mathrm{f}(x)$ has a stationary point when $x = a$, then:

- if $\mathrm{f}''(a) > 0$, the point is a local minimum
- if $\mathrm{f}''(a) < 0$, the point is a local maximum.

If $\mathrm{f}''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.

Differentiation 1

You can **differentiate** a function to find its **derivative** or **gradient function**.

The derivative is written as $f'(x)$ or $\frac{dy}{dx}$

Differentiating x^n

$y = f(x)$ $y = x^n$ — Multiply by the power …

Differentiation

$\frac{dy}{dx} = f'(x)$ $\frac{dy}{dx} = nx^{n-1}$ — … then reduce the power by 1

This rule works for **any** value of n, including **fractions** and **negative** numbers.

Golden rules

1. Write every term in a polynomial in the form ax^n **before** differentiating.
   $\sqrt{x} \to x^{\frac{1}{2}}$ $\frac{6}{x^2} \to 6x^{-2}$
2. Constant terms differentiate to **zero**, and x terms differentiate to a **constant**.

$f(x) = 7 \to f'(x) = 0$ $f(x) = 3x + 1 \to f'(x) = 3$

Worked example

Given that $y = 3x^6 - 8 + \frac{1}{x^3}$, $x \neq 0$, find $\frac{dy}{dx}$ **(3 marks)**

$y = 3x^6 - 8 + x^{-3}$

$\frac{dy}{dx} = 18x^5 - 3x^{-4}$

Start by rewriting $\frac{1}{x^3}$ as x^{-3}. Remember that you are multiplying by -3 when you differentiate this term, so the new term is **negative**.

Worked example

Differentiate with respect to x

(a) $x^3 - 3\sqrt{x} + \frac{x}{7}$ **(3 marks)**

$f(x) = x^3 - 3x^{\frac{1}{2}} + \frac{1}{7}x$

$f'(x) = 3x^2 - \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{7}$

(b) $\frac{kx + 5}{x^2}$ **(3 marks)**

$f(x) = \frac{kx}{x^2} + \frac{5}{x^2} = kx^{-1} + 5x^{-2}$

$f'(x) = -kx^{-2} - 10x^{-3}$

With respect to x just means that x is the variable. You should treat any other letters in the function as **constants**.

It's OK to leave powers as **negative numbers** or **fractions** in your final answers. The answer to part (a) could be in either of these forms:

$3x^2 - \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{7}$ $3x^2 - \frac{3}{2\sqrt{x}} + \frac{1}{7}$

Now try this

1 Given that $y = \frac{(x+3)^2}{x}$, $x \neq 0$, find $\frac{dy}{dx}$ Answer **(4 marks)**

Multiply out the brackets, then write the function in the form $y = ax + b + cx^{-1}$ before differentiating.

2 (a) Write $\frac{2 + 5\sqrt{x}}{x}$ in the form $2x^p + 5x^q$, where p and q are constants. **(2 marks)** Answer

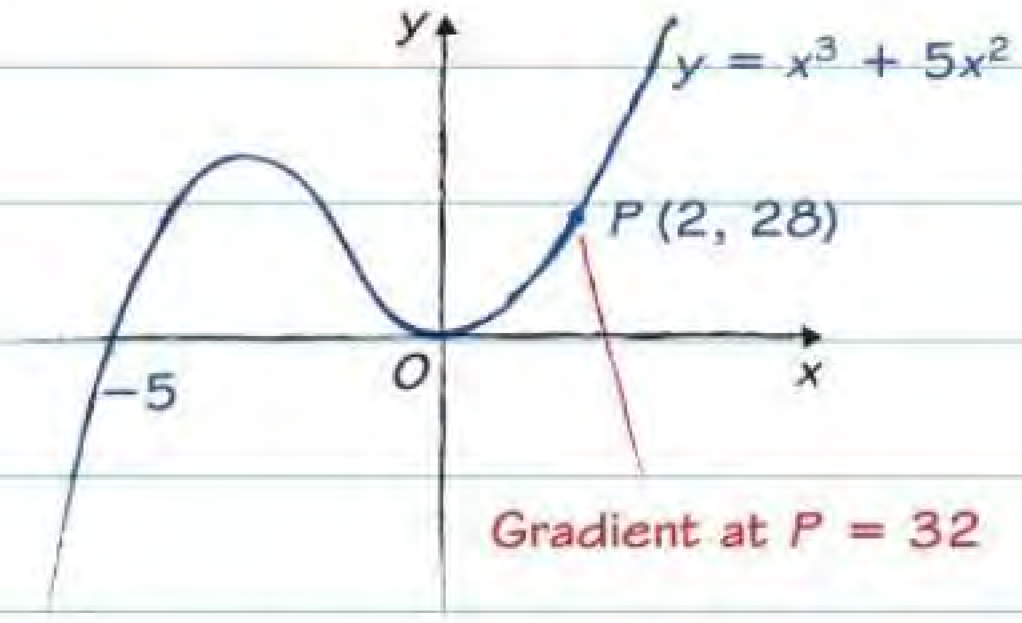
(b) Given that $y = 3x^2 + 1 - \frac{2 + 5\sqrt{x}}{x}$, find $\frac{dy}{dx}$ **(4 marks)** Answer

Had a look ☐ Nearly there ☐ Nailed it! ☐

Differentiation 2

You can use the **derivative** or **gradient function** to find the **rate of change** of a function, or the gradient of a curve.

This curve has equation $y = x^3 + 5x^2$. Its gradient function has equation $\frac{dy}{dx} = 3x^2 + 10x$. You can find the **gradient** at any point on the graph by substituting the x-coordinate at that point into the gradient function.



At the point P:

$x = 2$

$\frac{dy}{dx} = 3(2)^2 + 10(2) = 12 + 20 = 32$

Worked example

$f(x) = (10 + 2\sqrt{x})^2, x > 0$

(a) Show that $f(x) = 100 + k\sqrt{x} + 4x$, where k is a constant to be found. **(2 marks)**

$f(x) = 10^2 + 20\sqrt{x} + 20\sqrt{x} + (2\sqrt{x})^2$

$= 100 + 40\sqrt{x} + 4x \qquad k = 40$

(b) Find $f'(x)$. **(2 marks)**

$f(x) = 100 + 40x^{\frac{1}{2}} + 4x$

$f'(x) = 20x^{-\frac{1}{2}} + 4$

(c) Evaluate $f'(25)$. **(1 mark)**

$f'(25) = 20\left(25^{-\frac{1}{2}}\right) + 4$

$= 20\left(\frac{1}{5}\right) + 4$

$= 4 + 4 = 8$

Evaluating f′(x)

$f'(x)$ tells you the **rate of change** of the function for a given value of x.
You can calculate $f'(x)$ for a given value of x by substituting that value of x into the derivative.

$25^{-\frac{1}{2}} = \frac{1}{25^{\frac{1}{2}}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$

For a reminder about using the index laws to simplify powers have a look at page 1.

Second-order derivatives

You can differentiate **twice** to find the **second-order** derivative.
You write the second-order derivative as $\frac{d^2y}{dx^2}$ or $f''(x)$.

$y = 5x^3$

Differentiate

$\frac{dy}{dx} = 15x^2$

Differentiate

$\frac{d^2y}{dx^2} = 30x$

Worked example

Given that $y = 8\sqrt{x} - 3x^2 + 5x, x > 0$,

find $\frac{d^2y}{dx^2}$ **(4 marks)**

$y = 8x^{\frac{1}{2}} - 3x^2 + 5x$

$\frac{dy}{dx} = 4x^{-\frac{1}{2}} - 6x + 5$

$\frac{d^2y}{dx^2} = -2x^{-\frac{3}{2}} - 6$

Now try this

1 $f(x) = 3x^3 + 5x, x > 0$ Answer

(a) Differentiate to find $f'(x)$. **(2 marks)**

(b) Given that $f'(x) = 41$, find the value of x. **(3 marks)**

2 Given that $y = 2x^2 + 4x^{-2}$, find $\frac{d^2y}{dx^2}$ Answer **(4 marks)**

Answer

3 The curve C has equation $y = x(x - 1)(x + 3)$

(a) Find $\frac{dy}{dx}$ **(2 marks)** Answer

(b) Sketch C, showing each point where C meets the x-axis. **(3 marks)** Answer

(c) Find the gradient of C at each point where the curve meets the x-axis. **(2 marks)** Answer

Tangents and normals

You can use **differentiation** to work out the equations of tangents and normals.

The curve drawn in black has equation

$y = x^2 - 2x + 4$

You can differentiate this to work out the gradient function:

$\frac{dy}{dx} = 2x - 2$

There is more about finding gradient functions on page 36.

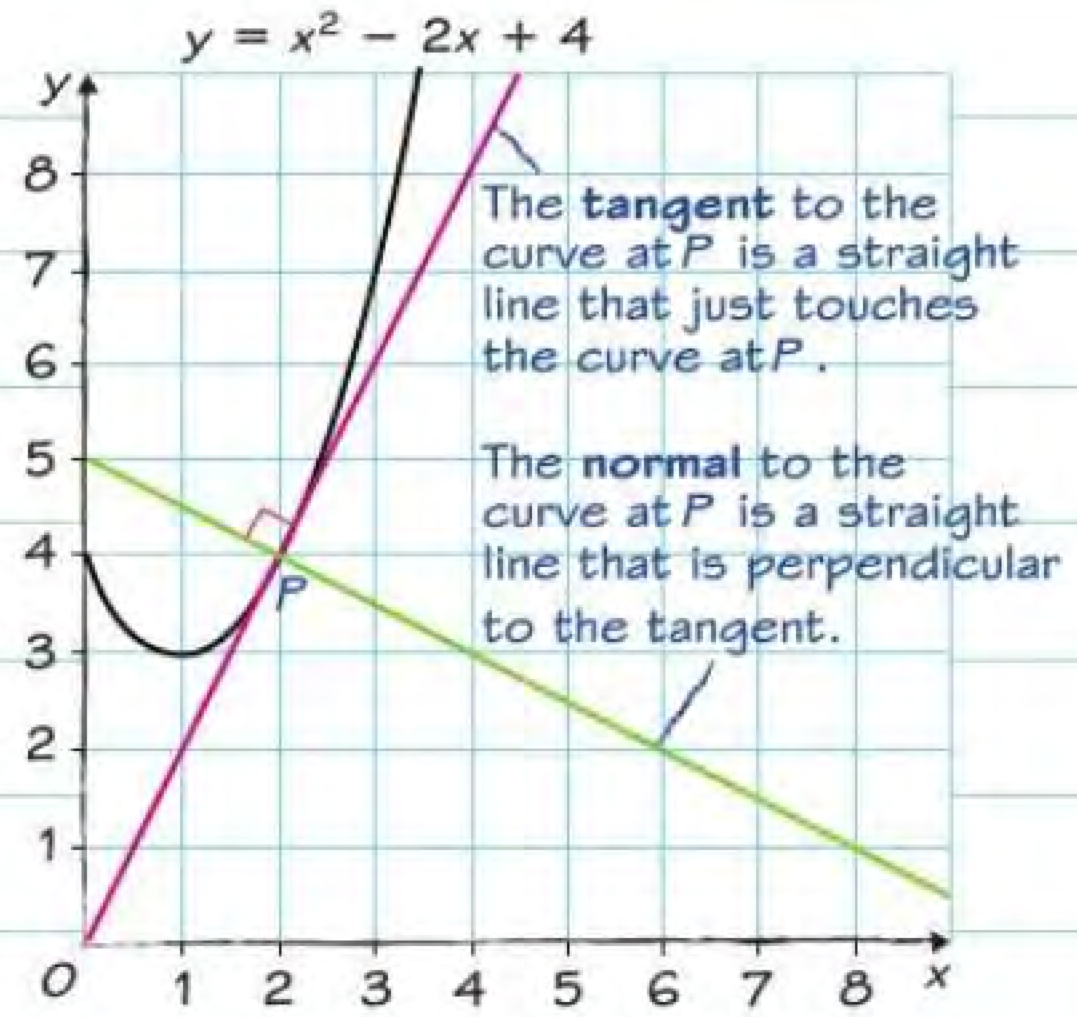
At the point P (2, 4) the gradient of the curve is 2, so the gradient of the tangent is also 2.

The tangent passes through (2, 4) and it has equation $y = 2x$.

The normal is perpendicular to the tangent, so it has gradient $-\frac{1}{2}$.

The normal also passes through (2, 4) and it has equation $y = -\frac{1}{2}x + 5$.

There is more about parallel and perpendicular lines on page 18.



Golden rules

If a curve has gradient m at point P:

1. the **tangent** at P **also** has gradient m
2. the **normal** at P has gradient $-\frac{1}{m}$

Worked example

The curve C has equation

$y = x^3 - 3x^2 - x + 9, \quad x > 0$

The point P with coordinates (2, 3) lies on C.

Find the equation of the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants. **(5 marks)**

$\frac{dy}{dx} = 3x^2 - 6x - 1$

$= 3(2)^2 - 6(2) - 1$

$= 12 - 12 - 1$

$= -1$

$y - y_1 = m(x - x_1)$

$y - 3 = (-1)(x - 2)$

$y - 3 = -x + 2$

$y = -x + 5$

Follow these steps:

1. Differentiate to find the gradient function.
2. Substitute $x = 2$ into $\frac{dy}{dx}$. The gradient at point P is -1.
3. Use $y - y_1 = m(x - x_1)$ to find the equation of a straight line with gradient -1 that passes through (2, 3).

There is more on finding equations of straight lines on page 17.

Now try this

The curve C has equation $y = f(x)$, $x > 0$, where

$\frac{dy}{dx} = \sqrt{x} + \frac{8}{x^2} - 5$

Given that the point P (4, 11) lies on C, find the equation of the normal to C at point P, giving your answer in the form $ax + by + c = 0$ **(4 marks)**

Answer

You have been given $\frac{dy}{dx}$, so start by substituting the x-coordinate of P to find the gradient of the **tangent** at P. The **normal** will be perpendicular to this.

Had a look ☐ Nearly there ☐ Nailed it! ☐

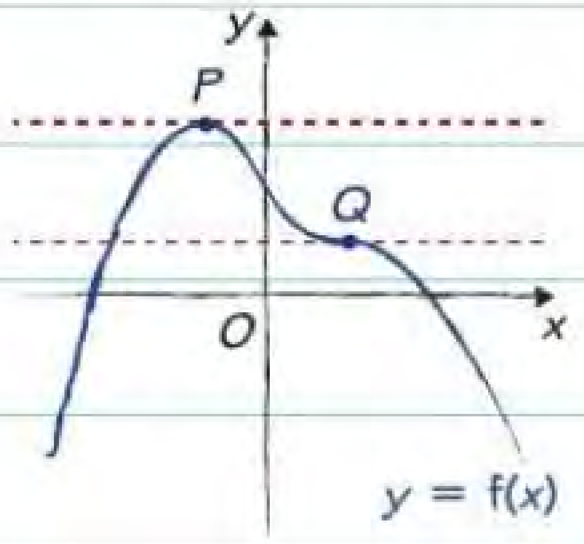
Stationary points 1

You can use **calculus** to find the stationary points of a **graph** or **function** in your exam. You need to be confident with **differentiation** – have a look at pages 36 and 37 for a reminder.

Using differentiation

The stationary points of a graph or function are the points where the **derivative**, $\frac{dy}{dx}$ or $f'(x)$, is equal to **zero**.

This graph has stationary points at P and Q. The slope of the curve is 0 at both points.



Worked example

Find the coordinates of the stationary point on the curve with equation $y = 3x^2 + 12x + 5$ **(4 marks)**

$\frac{dy}{dx} = 6x + 12$

When $\frac{dy}{dx} = 0$, $6x + 12 = 0$

$6x = -12$

$x = -2$

So $y = 3 \times (-2)^2 + 12 \times (-2) + 5 = -7$

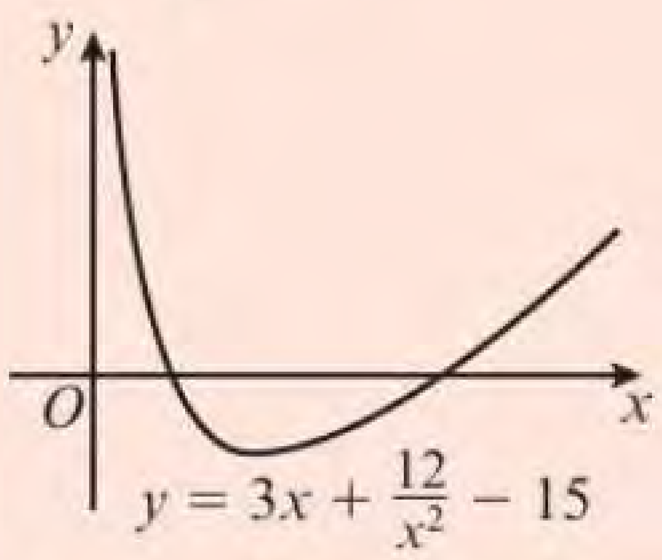
Stationary point is $(-2, -7)$.

To find the coordinates of the stationary point using calculus:

1. Differentiate to find $\frac{dy}{dx}$.
2. Set $\frac{dy}{dx} = 0$.
3. Solve the equation to find the value or values of x.
4. Find the corresponding value of y for each value of x.

Worked example

The diagram shows part of the curve with equation $y = 3x + \frac{12}{x^2} - 15$



Use calculus to show that y is increasing for $x > 2$ **(4 marks)**

$y = 3x + 12x^{-2} - 15$

$\frac{dy}{dx} = 3 - 24x^{-3} = 3 - \frac{24}{x^3}$

If $x > 2$ then $x^3 > 8$ and $\frac{24}{x^3} < 3$

So $3 - \frac{24}{x^3} > 0$

So if $x > 2$, $\frac{dy}{dx} \geq 0$ therefore y is increasing.

You need to show that the **derivative** is **non-negative** for all values of x in the range given.

Increasing and decreasing functions

You can use the derivative to decide if a function is increasing or decreasing in a given interval:

- If $f'(x) \geq 0$ for $a < x < b$ then $f(x)$ is INCREASING in the interval $a \leq x \leq b$.
- If $f'(x) \leq 0$ for $a < x < b$ then $f(x)$ is DECREASING in the interval $a \leq x \leq b$.

The sign of $f'(x)$ (+ or –) must be the same (or 0) in the whole interval, otherwise the function is neither decreasing nor increasing.

Now try this

1 Find the coordinates of the stationary point on the curve C with equation $y = x^2 - 8x + 3$ **(4 marks)** Answer

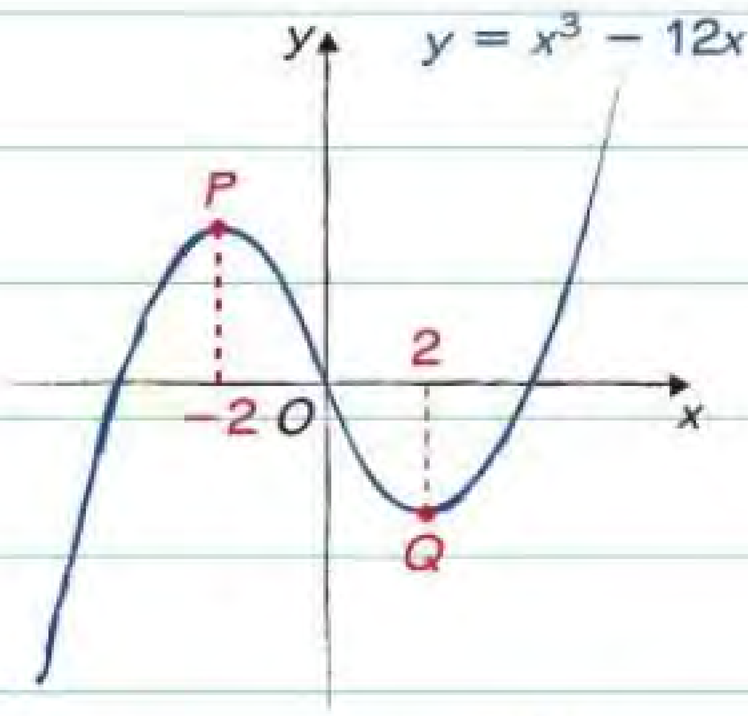
2 Use calculus to find the x-coordinates of the stationary points on the curve with equation $y = x^3 - 5x^2 + 8x + 1$ **(4 marks)** Answer

There are two stationary points. Differentiate, then solve a quadratic equation by factorising.

Stationary points 2

There are different types of stationary points on graphs. You need to be able to decide on the **nature** of a particular point. You can do this by finding the value of the **second derivative**, $\frac{d^2y}{dx^2}$ or $f''(x)$ at that point.

Maximum or minimum?



$\frac{dy}{dx} = 3x^2 - 12$ $\frac{d^2y}{dx^2} = 6x$

1 If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ then the stationary point is a **maximum**.

At P, $\frac{dy}{dx} = 3 \times (-2)^2 - 12 = 0$ and $\frac{d^2y}{dx^2} = 6 \times (-2) = -12 < 0$ so P is a **maximum**.

2 If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ then the stationary point is a **minimum**.

At Q, $\frac{dy}{dx} = 3 \times (2)^2 - 12 = 0$ and $\frac{d^2y}{dx^2} = 6 \times (2) = 12 > 0$ so Q is a **minimum**.

For a reminder about finding $\frac{d^2y}{dx^2}$ have a look at page 37.

Worked example

The curve C has equation $y = 12\sqrt{x} - 2x$, $x > 0$

(a) Use calculus to find the coordinates of the turning point of C. **(4 marks)**

$y = 12x^{\frac{1}{2}} - 2x$

$\frac{dy}{dx} = 6x^{-\frac{1}{2}} - 2 = \frac{6}{\sqrt{x}} - 2$

When $\frac{dy}{dx} = 0$, $\frac{6}{\sqrt{x}} - 2 = 0$

$6 = 2\sqrt{x}$

$x = 9$

So $y = 12\sqrt{9} - 2 \times 9 = 18$

Turning point is (9, 18).

(b) Find $\frac{d^2y}{dx^2}$ **(2 marks)**

$\frac{d^2y}{dx^2} = -3x^{-\frac{3}{2}}$

(c) State the nature of the turning point. **(1 mark)**

At turning point, $x = 9$, so $\frac{d^2y}{dx^2} = -3 \times 9^{-\frac{3}{2}}$

$= -\frac{1}{9}$

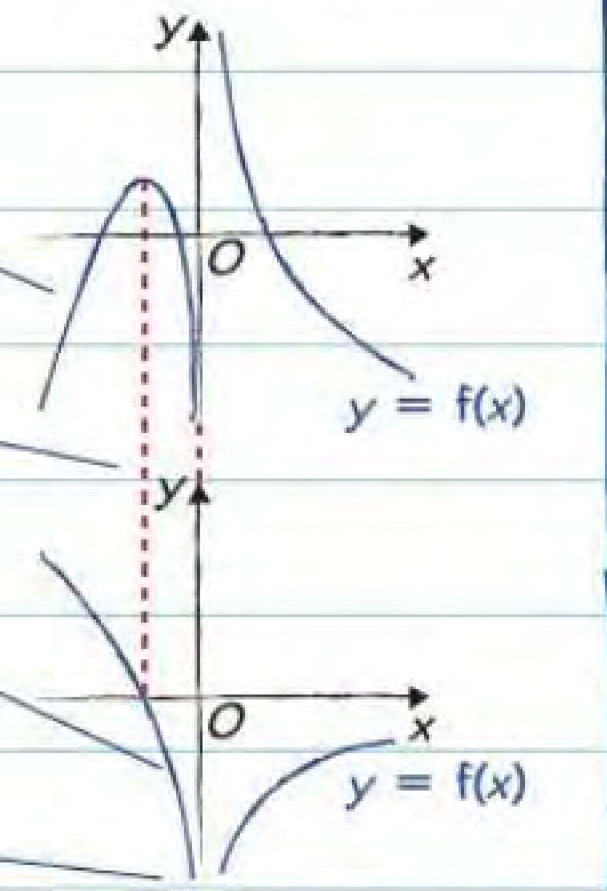
$\frac{d^2y}{dx^2} < 0$ so the turning point is a maximum.

A turning point is a stationary point which is a **local maximum** or a **local minimum**.

For part (c), work out the value of $\frac{d^2y}{dx^2}$ when $x = 9$. Write down whether it is greater or less than 0 and state whether the turning point is a maximum or a minimum.

Sketching gradient functions

You can use the features of $y = f(x)$ to sketch $y = f'(x)$.



Now try this

The diagram shows a sketch of the curve with equation $y = 5x^2 - 3x - x^3$

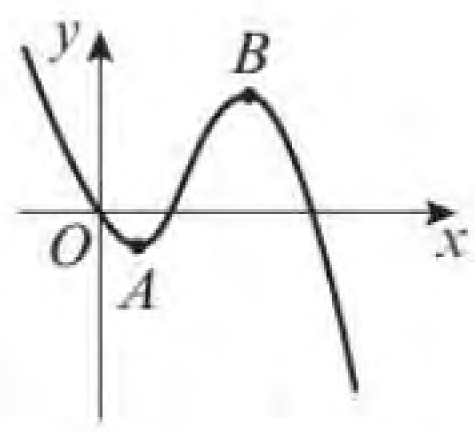
The curve has stationary points at A and B.

Answer

(a) Use calculus to find the coordinates of A and B. **(6 marks)**

(b) Find the value of $\frac{d^2y}{dx^2}$ at B, and hence verify that B is a maximum. **(2 marks)**

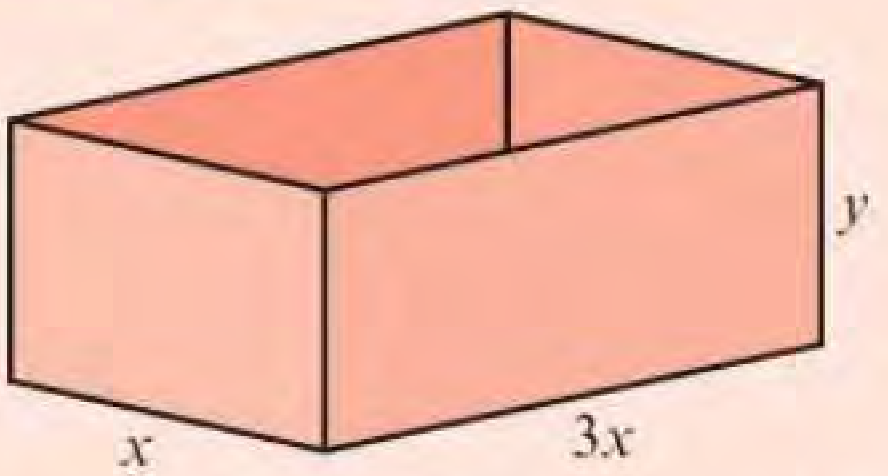
Answer



Modelling with calculus

You can use **calculus** to solve real-life problems involving maximums and minimums.

Worked example



A cardboard box is made in the shape of an open-topped cuboid, with volume 18 000 cm^3. The base of the cuboid has width x cm and length $3x$ cm. The height of the cuboid is y cm.

(a) Show that the area, A cm^2, of cardboard needed to make the box is given by

$$A = 3x^2 + \frac{48\,000}{x}$$ **(4 marks)**

Volume $= 3x^2y = 18\,000$

$y = \frac{6000}{x^2}$

$A = 3x^2 + 2 \times xy + 2 \times 3xy$

$= 3x^2 + 8xy$

$= 3x^2 + 8x\left(\frac{6000}{x^2}\right)$

$= 3x^2 + \frac{48\,000}{x}$

(b) Use calculus to find the value of x for which A is stationary. **(4 marks)**

$\frac{dA}{dx} = 6x - \frac{48\,000}{x^2}$

When $\frac{dA}{dx} = 0$, $6x - \frac{48\,000}{x^2} = 0$

$6x^3 = 48\,000$

$x^3 = 8000$

$x = 20$

(c) Show that A is a minimum at this point and find its value. **(4 marks)**

$\frac{d^2A}{dx^2} = 6 + \frac{96\,000}{x^3}$

When $x = 20$, $\frac{d^2A}{dx^2} = 6 + \frac{96\,000}{20^3} = 18 > 0$

So A is a minimum.

$A = 3x^2 + \frac{48\,000}{x} = 3 \times (20)^2 + \frac{48\,000}{20}$

$= 3600$

This is the least area of carboard needed to make the box.

Problem solved!

(a) Use the information given about the volume of the cuboid to write y in terms of x. Then write A in terms of x and y, and substitute your first expression to get A in terms of x only.

(b) Because A is a function of x only, you can differentiate with respect to x to find $\frac{dA}{dx}$. Find the stationary point of A by setting this equal to 0 and solving to find x.

(c) You need to find $\frac{d^2A}{dx^2}$ to determine the nature of the stationary point.

You will need to use problem-solving skills throughout your exam – **be prepared!**

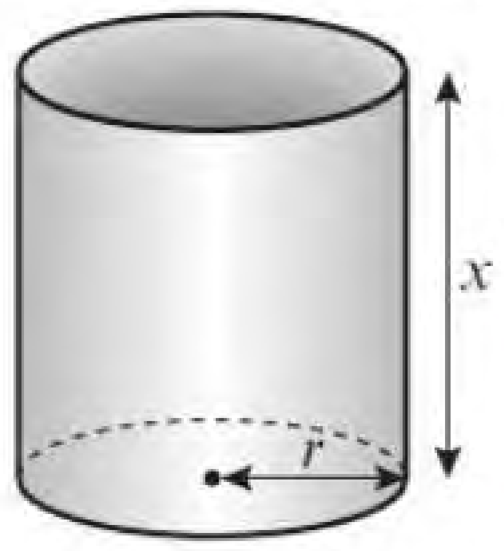
Now try this

1 An oil well produces x barrels of oil each day. It models its profit, £P each day, using the formula $P = 80x - \frac{x^2}{50}$

(a) Find $\frac{dP}{dx}$ **(2 marks)** Answer

(b) Hence show that P has a stationary point at $x = 2000$ and use calculus to determine the nature of that stationary point. **(4 marks)** Answer

2 The diagram shows a container in the shape of an open-topped cylinder, with height x m and radius r m. The cylinder has a capacity of 100 m^3.



(a) Show that the area of sheet metal, A m^2, needed to make the tank is given by $A = \pi r^2 + \frac{200}{r}$ **(4 marks)** Answer

(b) Use calculus to find the value of r for which A is stationary. **(4 marks)** Answer

(c) Prove that this value of r gives a minimum value of A. **(2 marks)** Answer

(d) Hence calculate the minimum area of sheet metal needed to make the tank. **(2 marks)** Answer

When you differentiate, remember that π is constant