Summary of key points

1 If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

2 If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

When integrating polynomials, apply the rule of integration separately to each term.

3 $\int f'(x)dx = f(x) + c$

4 $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

5 To find the constant of integration, c

- Integrate the function
- Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$ into the integrated function
- Solve the equation to find c

Integration

Integration is the **opposite** of differentiation. You can use this rule to integrate terms which are written in the form ax^n.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

- This is the symbol for integration.
- You **increase** the power by 1 …
- … then **divide** by the **new** power.
- This rule **doesn't** work if the original power is −1
- You are integrating **with respect to** x.
- You have to add the **constant of integration**.

To **integrate** a function, write each term in the form ax^n, then integrate one term at a time.

The constant of integration

When you **differentiate**, any **constant terms** disappear. So lots of functions have the same derivative.

$y = x^2 + 5$

$y = x^2$ **Differentiate** $\frac{dy}{dx} = 2x$ **Integrate** $y = x^2 + c$

$y = x^2 - 19$

When you integrate you don't know the constant. You write '$+ c$' at the end to show this. This is called **indefinite integration**.

Worked example

Find $\int(12x^3 + 6x - 15x^{\frac{2}{3}})\,dx$, giving each term in its simplest form. **(5 marks)**

$\int\left(12x^3 + 6x - 15x^{\frac{2}{3}}\right) dx$

$= \frac{12x^4}{4} + \frac{6x^2}{2} - \frac{15x^{\frac{5}{3}}}{\left(\frac{5}{3}\right)} + c$

$= 3x^4 + 3x^2 - 9x^{\frac{5}{3}} + c$

Integrate term-by-term and don't forget to add the constant of integration.
For each term:
- increase the power by 1
- divide by the **new** power.

$\frac{2}{3} + 1 = \frac{5}{3}$. Dividing by $\frac{5}{3}$ is the same as dividing by 5 then multiplying by 3.

Golden rules

1. Write every term in a polynomial in the form ax^n **before** integrating.
2. Remember to include the **constant of integration**.
3. Simplify any **coefficients** if possible.

Worked example

Given that $y = \frac{1}{x^3} - 3x^5$, $x \neq 0$, find $\int y\,dx$ **(3 marks)**

$\int(x^{-3} - 3x^5)\,dx = \frac{x^{-2}}{-2} - \frac{3x^6}{6} + c$

$= -\frac{1}{2}x^{-2} - \frac{1}{2}x^6 + c$

Be careful with negative powers. For the first term, you have to **increase** the power of −3 by 1 to get −2, then divide by the **new power**, −2.

Now try this

1 Find $\int(1 - 3x^3)\,dx$ Answer **(3 marks)**

2 Find $\int(3x + 1)^2\,dx$ Answer **(4 marks)**

Expand the brackets first.

3 Given that $y = 6x^2 + 5x\sqrt{x}$, $x > 0$, find $\int y\,dx$ Answer **(3 marks)**

Finding the constant

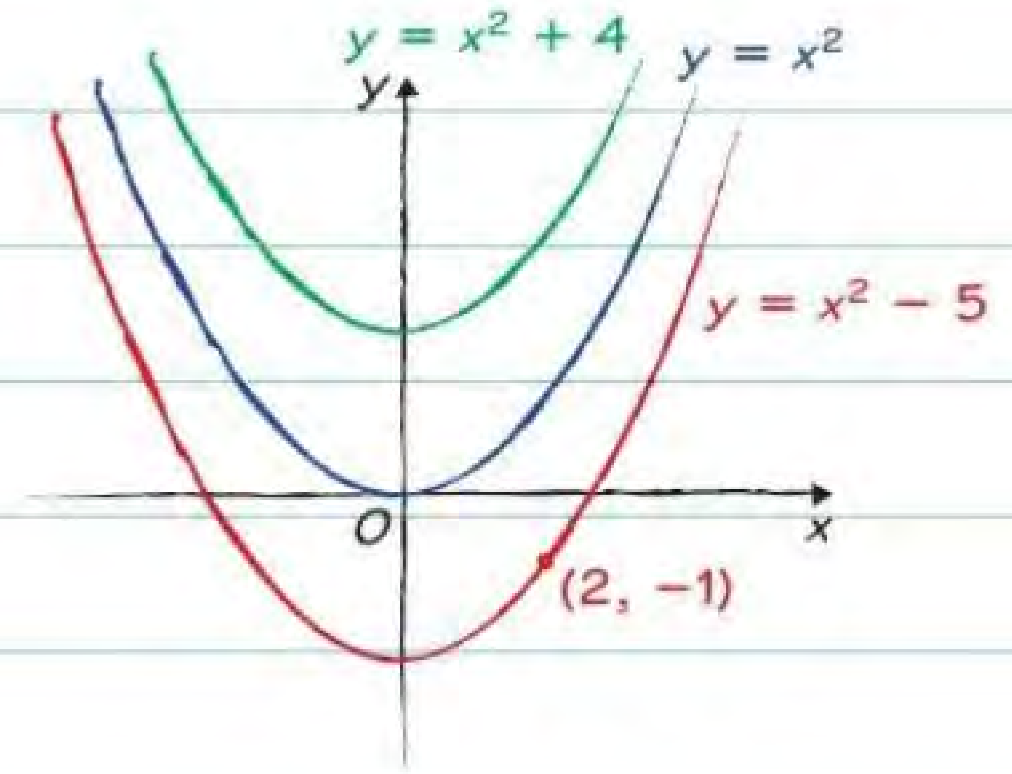
If you know **one point** on the original curve, or **one value** of $f(x)$, then you can calculate the constant of integration.

Using substitution

All three of these curves have the same gradient function:

$$\frac{dy}{dx} = 2x$$

But only one passes through the point $(2, -1)$. You can find its equation by integrating, then substituting $x = 2$ and $y = -1$ to find the value of c.



$y = x^2 + c$
$-1 = 2^2 + c$
$c = -5$
$y = x^2 - 5$

Worked example

$\frac{dy}{dx} = 7 + \frac{1}{\sqrt{x}}, x > 0$

Given that $y = 26$ at $x = 4$, find y in terms of x. **(6 marks)**

$\frac{dy}{dx} = 7 + x^{-\frac{1}{2}}$

$y = 7x + \frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$

$= 7x + 2x^{\frac{1}{2}} + c$

$26 = 7(4) + 2(4)^{\frac{1}{2}} + c$

$= 28 + 4 + c$

$c = -6$

$y = 7x + 2x^{\frac{1}{2}} - 6$

Worked example

Given that $f(-1) = 9$ and $f'(x) = 6x^2 - 10x - 3$, find $f(x)$. **(5 marks)**

$f(x) = \frac{6x^3}{3} - \frac{10x^2}{2} - 3x + c$

$= 2x^3 - 5x^2 - 3x + c$

$f(-1) = 2(-1)^3 - 5(-1)^2 - 3(-1) + c$

$= -2 - 5 + 3 + c$

$= -4 + c$

$9 = -4 + c$

$c = 13$

$f(x) = 2x^3 - 5x^2 - 3x + 13$

$f(-1) = 9$ is the same as saying that the curve with equation $y = f(x)$ passes through the point $(-1, 9)$.

Three key steps

1. Integrate, and remember to include the constant of integration.
2. Substitute the values of x and y you know and solve an equation to find c.
3. Write out the function including the constant of integration you've found.

Now try this

1 The curve C has equation $y = f(x)$, $x > 0$, and the point $(4, 17)$ lies on C.

Given that $f'(x) = 3 - \frac{2 + 3\sqrt{x}}{x^2}$, find $f(x)$. **(5 marks)**

Answer

2 $\frac{dy}{dx} = \frac{(x^2 + 5)^2}{x^2}, x \neq 0$

(a) Show that $\frac{dy}{dx} = x^2 + 10 + 25x^{-2}$ **(2 marks)** Answer

(b) Given that $y = -13$ at $x = 1$, find y in terms of x. **(6 marks)** Answer