Summary of key points

1 To prove a statement by contradiction you start by assuming it is **not true**. You then use logical steps to show that this assumption leads to something impossible (either a contradiction of the assumption or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement **was true**.

2 A rational number can be written as $\frac{a}{b}$, where a and b are integers.

An irrational number cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.

Proof by contradiction

You can use the following steps to prove something by contradiction:

1 Assume... ...the **negation** of the original statement.

For example, to prove that there is no largest odd number, assume that there exists a largest odd number, n.

2 Use... ...logical steps to show that this assumption leads to a **contradiction**.

If n is an odd number then $n + 2$ is also an odd number. You also know that $n + 2 > n$.

3 Conclude... ...that the assumption was **incorrect** and the original statement is **true**.

This contradicts the assumption that n is the largest odd number. You can conclude that there is **no** largest odd number.

Negation

The first step in a proof by contradiction is to assume the negation of a statement. The negation of a statement is a statement that asserts its falsehood.

Statement	Negation
All mice are white	There exists a mouse that is not white
There are infinitely many prime numbers	There are finitely many prime numbers
There is no smallest rational number	There exists some smallest rational number, n
$\sqrt{2}$ is an irrational number	$\sqrt{2}$ is a rational number

Rational or irrational?

You might have to use contradiction to prove statements involving rational numbers.

- A **rational number** is a number that can be written in the form $\frac{a}{b}$, where a and b are integers.

  Examples are 4, $\frac{2}{3}$, $-\frac{12}{7}$ and 0.

- An **irrational number** is a number that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.

  You can use proof by contradiction to show that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

Worked example

Prove by contradiction that if $p + q$ is an irrational number, then at least one of p and q is an irrational number. **(3 marks)**

Assumption: $p + q$ is an irrational number and both p and q are rational numbers.

Write $p = \frac{a}{b}$ and $q = \frac{c}{d}$, where a, b, c and d are all integers.

Then $p + q = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

Since a, b, c and d are all integers, $\frac{ad + bc}{bd}$ must be rational, and hence $p + q$ is rational. ※

Therefore at least one of p and q must be irrational.

Problem solved!

When completing a proof by contradiction, you should always state your assumption. Use the word 'assumption' or 'assume' to show that you understand the steps of the proof.

If you have assumed that a number is rational, then a good starting point is to write it in the form $\frac{a}{b}$ where a and b are integers. In some cases you might also need to assume that a and b have no common factors.

You will need to use problem-solving skills throughout your exam – **be prepared!**

You can use the symbol ※ to indicate a contradiction. You could also write out 'this contradicts the assumption that $p + q$ is an irrational number'.

Now try this

1 Prove that if p is a non-zero rational number and q is an irrational number, then pq is an irrational number. **(3 marks)**

Answer

2 Prove that there is no possible value θ for which 2, $\sin\theta$ and $\tan\theta$ are three consecutive terms in a geometric sequence. **(3 marks)**

Answer

See page 69 for a definition of a geometric sequence.