Summary of key points

3 To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.

4 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

5 To add or subtract two fractions, find a common denominator.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Algebraic fractions

You will usually be able to simplify algebraic fractions in your exam using these steps:

1 Factorise... ...expressions in the numerators or denominators into linear factors.

2 Cancel... ...common factors in the numerator and denominator of each fraction.

3 Add or subtract... ...any fractions by finding a common denominator, to make a **single fraction**.

If you **factorise** and **cancel** first then your fractions will be easier to **add or subtract**.

You might need to do steps 1 and 2 again once you have added or subtracted your fractions.

Common denominators

To add or subtract algebraic fractions with different denominators you need to find a common denominator. Here are two examples:

1. $$\frac{3}{x} + \frac{5}{2x+1} = \frac{3(2x+1)}{x(2x+1)} + \frac{5x}{x(2x+1)} = \frac{3(2x+1)+5x}{x(2x+1)}$$

The common denominator is the **product** of the two denominators.

2. $$\frac{2}{x+1} - \frac{2x}{(x+1)(x-2)} = \frac{2(x-2)}{(x+1)(x-2)} - \frac{2x}{(x+1)(x-2)} = \frac{2(x-2)-2x}{(x+1)(x-2)}$$

Once you are confident you might be able to skip the step shown in green above.

The denominators already **share** a factor of $(x + 1)$, so you only have to change the first fraction.

After this step, you can simplify the fractions further by expanding the brackets in the numerator and collecting like terms.

Worked example

Express $\frac{2(3x+2)}{9x^2-4} - \frac{2}{3x+1}$

as a single fraction in its simplest form. **(4 marks)**

$$\frac{2(3x+2)}{9x^2-4} - \frac{2}{3x+1} = \frac{2\cancel{(3x+2)}}{\cancel{(3x+2)}(3x-2)} - \frac{2}{3x+1}$$

$$= \frac{2}{3x-2} - \frac{2}{3x+1}$$

$$= \frac{2(3x+1) - 2(3x-2)}{(3x-2)(3x+1)}$$

$$= \frac{6}{(3x-2)(3x+1)}$$

You will probably have to do **multiple steps** so get used to writing out your working neatly. You can show which factors you are **cancelling** by drawing a neat line through them.

After you have added or subtracted your fractions you should expand the brackets in the numerator and simplify again:

$$2(3x+1) - 2(3x-2) = \cancel{6x} + 2 - \cancel{6x} + 4 = 6$$

You can leave the final denominator factorised like this, or multiply it out.

Now try this

1 Simplify fully $\frac{3x^2-8x-3}{x^2-9}$ **(3 marks)**

Answer

Watch out for the difference of two squares. Use $a^2 - b^2 = (a+b)(a-b)$

2 Express $\frac{x+5}{2x^2+7x-4} - \frac{1}{2x-1}$

as a single fraction in its simplest form. **(4 marks)**

Answer

Summary of key points

6 A single fraction with two distinct linear factors in the denominator can be split into two separate fractions with linear denominators. This is called splitting it into **partial fractions**:

$$\frac{5}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

7 The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator:

$$\frac{7}{(x-2)(x+6)(x+3)} = \frac{A}{x-2} + \frac{B}{x+6} + \frac{C}{x+3}$$

8 A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions:

$$\frac{2x+9}{(x-5)(x+3)^2} = \frac{A}{x-5} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

9 An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.

10 You can either use:

- algebraic division
- or the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$

to convert an improper fraction into a mixed fraction.

Partial fractions

Many algebraic fractions can be written as the **sum** of simpler fractions. This technique is called writing a fraction in **partial fractions**. In your A-level exam you might have to use partial fractions with the **binomial expansion** or when **integrating**. You can revise these topics on pages 75 and 105.

Worked example

$$f(x) = \frac{7 - 2x}{(2x - 1)(x + 1)}$$

Express f(x) in partial fractions. **(3 marks)**

$$\frac{7 - 2x}{(2x - 1)(x + 1)} = \frac{A}{2x - 1} + \frac{B}{x + 1}$$

$$7 - 2x = A(x + 1) + B(2x - 1)$$

Let $x = \frac{1}{2}$: $\quad 7 - 2\left(\frac{1}{2}\right) = A\left(\frac{1}{2} + 1\right)$

$\underline{A = 4}$

Let $x = -1$: $\quad 7 - 2(-1) = B(2(-1) - 1)$

$\underline{B = -3}$

$$f(x) = \frac{4}{2x - 1} - \frac{3}{x + 1}$$

Golden rule

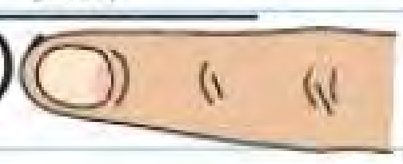
Find as many missing values as possible by **substituting** values for x to make some of the factors equal to zero. The more factors you can find this way, the easier it will be to **equate coefficients** later.

The denominators on the right-hand side are factors of the original denominator. If all the factors are different, then each one appears as a denominator once.

Cover up and calculate

If the expression has **no repeated factors**, you can use this quick method to find numerators. Choose a factor, and work out the value of x which makes that factor equal to zero. Then cover it up with your finger, and evaluate what's left of the fraction with that value of x.

$$f(x) = \frac{7 - 2x}{(2x - 1)\ \ }\qquad \frac{7 - 2(-1)}{2(-1) - 1} = \frac{9}{-3} = -3$$



Covering up $(x + 1)$ in the Worked example above and evaluating what's left with $x = -1$ gives you B.

Problem solved!

You need to do a bit more work if there is a repeated factor, like $(x - 3)^2$, or if the fraction is improper. You should always work out any values you can by substituting first. Here you can work out one value by substituting $x = 3$. To work out the other values you need to **equate coefficients**. You could multiply out both sides first:

$$2x^2 - 1 = Ax^2 + (-6A + B)x + (9A - 3B + C)$$

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

$$\frac{2x^2 - 1}{(x - 3)^2} = A + \frac{B}{(x - 3)} + \frac{C}{(x - 3)^2}, \quad x \neq 3$$

Find the values of A, B and C. **(4 marks)**

$2x^2 - 1 = A(x - 3)^2 + B(x - 3) + C$

Let $x = 3$: $\quad 2(3)^2 - 1 = C$

$\underline{C = 17}$

Equate x^2 terms: $\quad \underline{A = 2}$

Equate constant terms: $\quad -1 = 9A - 3B + C$

$-1 = 18 - 3B + 17$

$-36 = -3B$

$\underline{B = 12}$

Now try this

There is a repeated factor, so you need one fraction with denominator $(x + 2)$ and another fraction with denominator $(x + 2)^2$.

1 $g(x) = \dfrac{8x^2}{(3x - 2)(x + 2)^2}$

Express g(x) in partial fractions. **(4 marks)**

Answer

2 $\dfrac{6x^2 - 1}{(2x - 3)(x + 1)} = A + \dfrac{B}{2x - 3} + \dfrac{C}{x + 1}$

Find the values of A, B and C. **(4 marks)**

Answer

Algebraic division

You might need to find missing coefficients when a **cubic** or **quartic** expression is divided by a **quadratic** expression. You can use **long division**, but make sure you set your work out neatly.

Here is the working for $\dfrac{3x^4 - 6x^3 + x - 2}{x^2 - 1}$:

$$
\begin{array}{r|l}
 & 3x^2 - 6x + 3 \\
x^2 + 0x - 1 & 3x^4 - 6x^3 + 0x^2 + x - 2 \\
 & 3x^4 + 0x^3 - 3x^2 \\
 & \quad -6x^3 + 3x^2 + x - 2 \\
 & \quad -6x^3 + 0x^2 - 6x \\
 & \qquad 3x^2 - 5x - 2 \\
 & \qquad 3x^2 + 0x - 3 \\
 & \qquad\quad -5x + 1
\end{array}
$$

- You need to multiply $(x^2 - 1)$ by $3x^2$ to get the term $3x^4$, so the first term in your answer is $3x^2$
- The x coefficient is 0, so write $+ 0x$
- The x^2 coefficient is 0, so write $+ 0x^2$
- $3x^2 \times (x^2 + 0x - 1) = 3x^4 + 0x^3 - 3x^2$
- Always line up terms with the same power of x.
- Be careful with negative terms when subtracting: $-2 - (-3) = 1$
- If you are dividing by a quadratic, the remainder will be a linear term.

So $\dfrac{3x^4 - 6x^3 + x - 2}{x^2 - 1} = (3x^2 - 6x + 3) + \dfrac{1 - 5x}{x^2 - 1}$

(Quotient: $3x^2 - 6x + 3$; Divisor: $x^2 - 1$)

Worked example

Given that

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 4}, \quad x \neq \pm 2$$

find the values of the constants a, b, c, d and e. **(4 marks)**

$3x^4 - 2x^3 - 5x^2 - 4 \equiv (ax^2 + bx + c)(x^2 - 4) + dx + e$

$\equiv ax^4 + bx^3 + cx^2 - 4ax^2 - 4bx - 4c + dx + e$

$\equiv ax^4 + bx^3 + (c - 4a)x^2 + (d - 4b)x + (e - 4c)$

x^4 terms $\rightarrow$ $\underline{a = 3}$

x^3 terms $\rightarrow$ $\underline{b = -2}$

x^2 terms $\rightarrow$ $c - 4a = -5$

$c - 12 = -5$

$\underline{c = 7}$

x terms $\rightarrow$ $d - 4b = 0$

$d + 8 = 0$

$\underline{d = -8}$

Constant terms $\rightarrow$ $e - 4c = -4$

$e - 28 = -4$

$\underline{e = 24}$

You can also **compare coefficients** to find the missing coefficients. Follow these steps:

1. Multiply both sides by the divisor.
2. Expand the brackets **carefully** then collect like terms.
3. Compare coefficients on both sides, starting with the highest power of x.

As long as you write down what each constant is equal to, you don't need to write out the whole expression at the end.

Now try this

Given that

$$\frac{2x^4 + 4x^2 - x + 2}{x^2 - 1} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 1}, \quad x \neq \pm 1$$

find the values of the constants a, b, c, d and e.

Answer **(4 marks)**

Whichever method you use, make sure you either:

- write out the expression in full with the constants in place, or
- write $a = \ldots$, $b = \ldots$, etc.