## Summary of key points

**1** A modulus function is, in general, a function of the type $y = |f(x)|$.

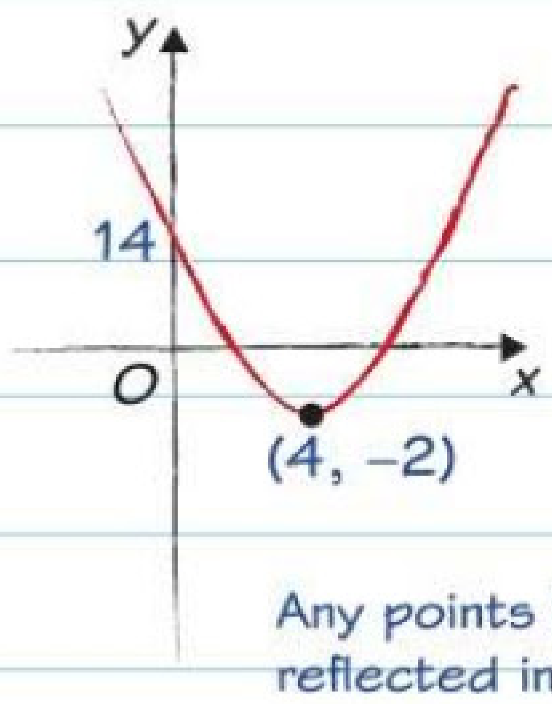
- When $f(x) \geqslant 0$, $|f(x)| = f(x)$
- When $f(x) < 0$, $|f(x)| = -f(x)$

**2** To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the $x$-axis in the $x$-axis.

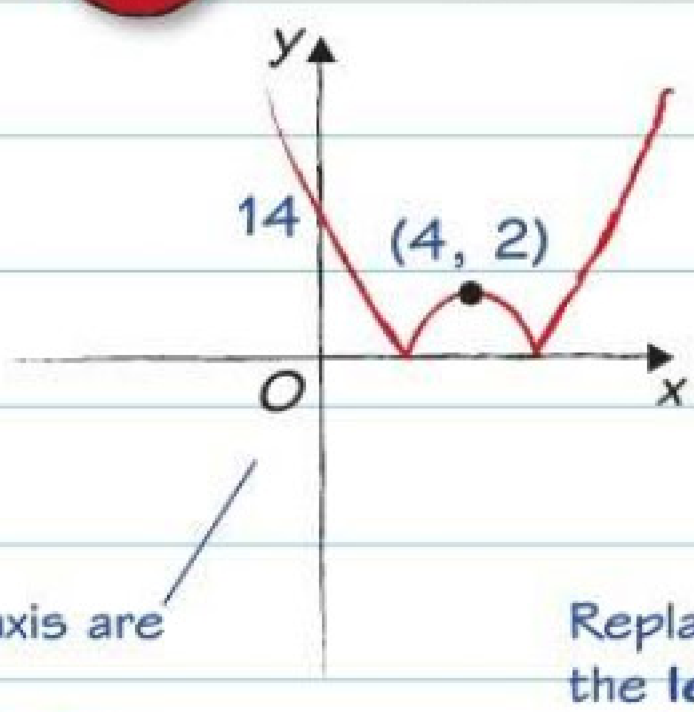
# Modulus

The **modulus** of a number is its positive numerical value. You write the modulus of $x$ as $|x|$.
You can use the graph of $y = f(x)$ to sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.

$y = f(x)$

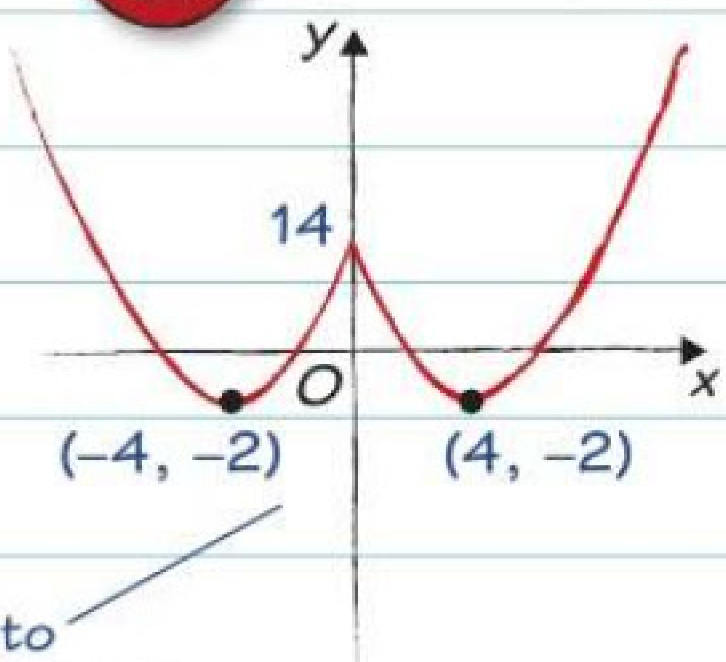



**1** $y = |f(x)|$




Any points **below** the $x$-axis are reflected in the $x$-axis.
Every point on the curve must have a **non-negative** $y$-coordinate.

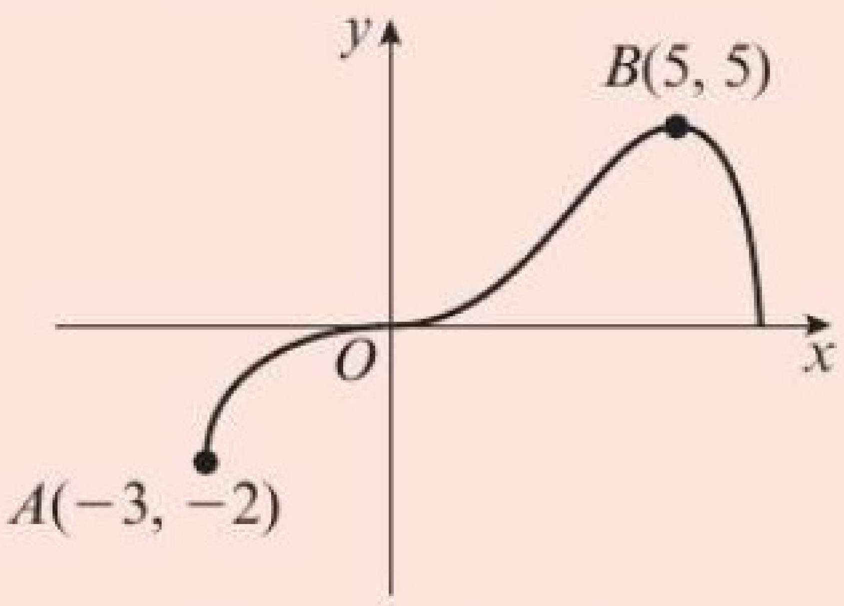
**2** $y = f(|x|)$




Replace the curve to the **left** of the $y$-axis with a reflection of the curve to the **right** of the $y$-axis.

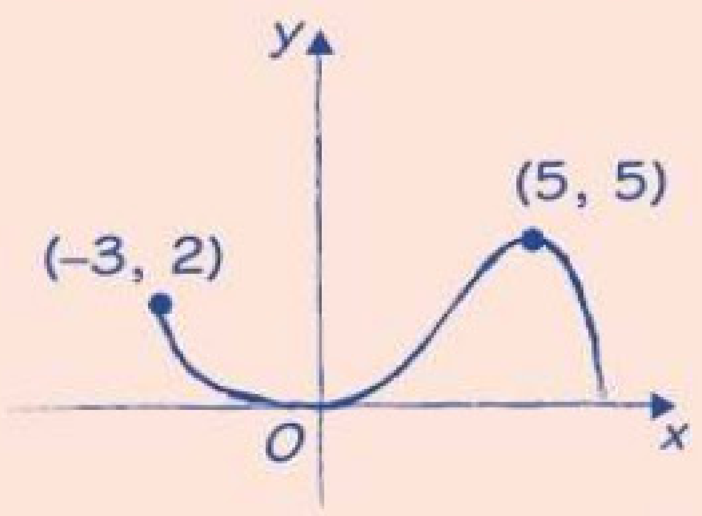
## Worked example

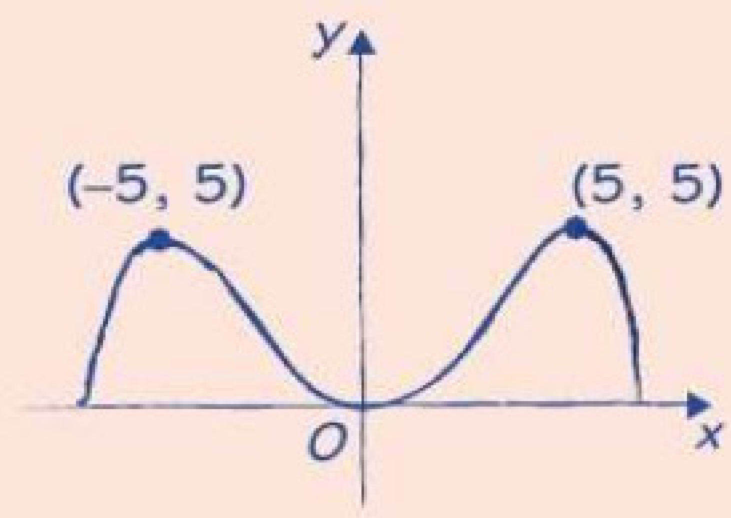
The diagram shows a sketch of the curve with equation $y = f(x)$.




On separate diagrams sketch the following graphs, showing the coordinates of the points corresponding to $A$ and $B$.

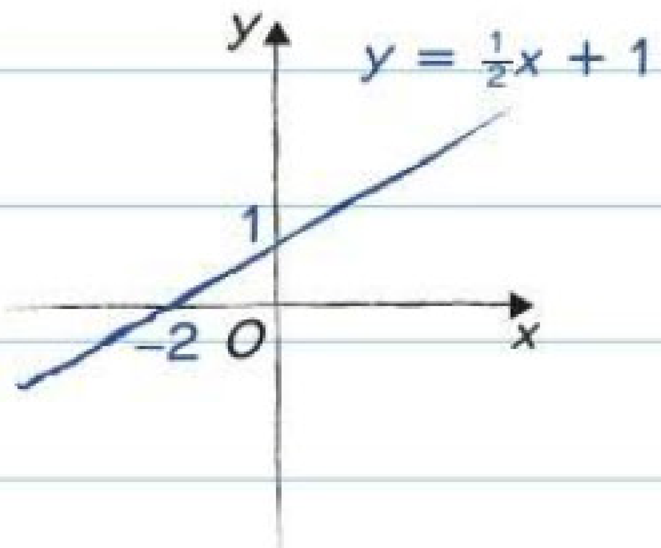
(a) $y = |f(x)|$ **(3 marks)** (b) $y = f(|x|)$ **(3 marks)**

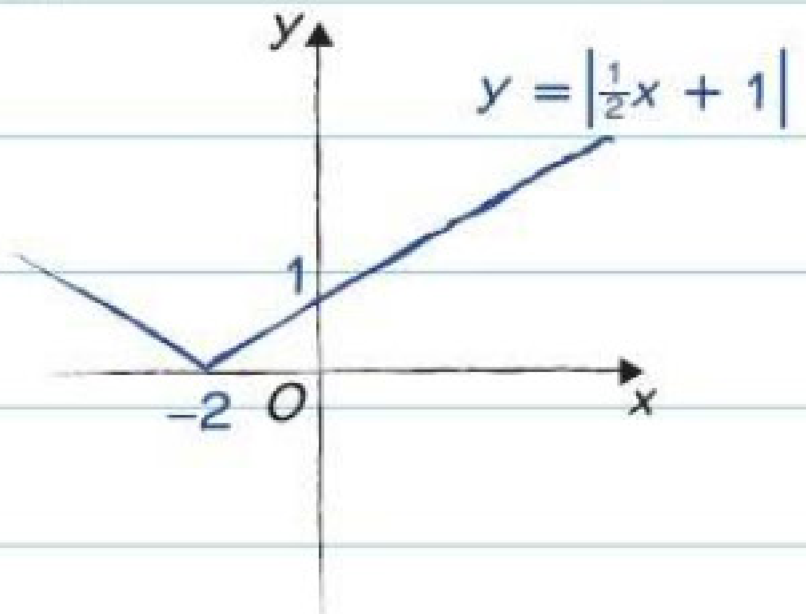



## Sketching $y = |ax + b|$

You can sketch the modulus of a linear function by sketching the graph, then reflecting any points that are below the $x$-axis.




The graph of $y = |ax + b|$ is always a V-shape.

## Worked example

The function f is defined by

$f : x \mapsto 3|x| - 5, \quad x \in \mathbb{R}$

State the range of f. **(2 marks)**

$f(x) \geqslant -5$

$|x|$ is always greater than or equal to 0.

## Now try this

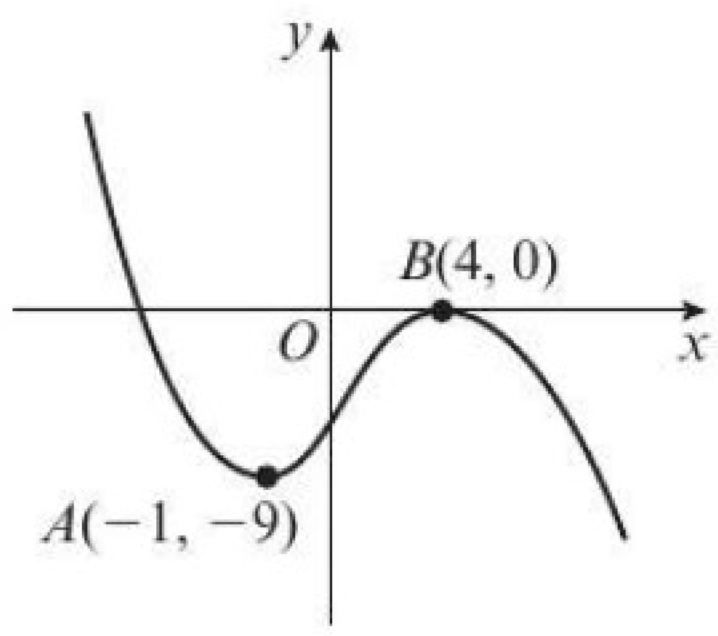
The diagram shows a sketch of the curve with equation $y = f(x)$.

On separate diagrams sketch the following graphs.

(a) $y = |f(x)|$ Answer **(3 marks)**

(b) $y = f(|x|)$ Answer **(3 marks)**

In each case, show the coordinates corresponding to the turning points $A$ and $B$.



# Modulus equations

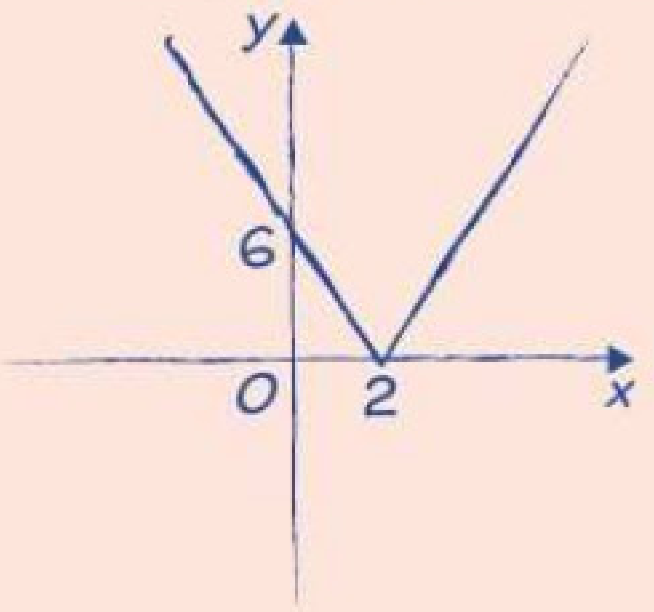
Solving an equation involving a modulus function is a bit like solving two equations. You need to consider the situations when the argument (the part inside the modulus) is **positive** and **negative** separately. You can use a graph to check that your answers make sense.

## Worked example

The function f is defined by

$f: x \mapsto |3x - 6|, \quad x \in \mathbb{R}$

(a) Sketch the graph with equation $y = f(x)$, showing the coordinates of the points where the graph cuts or meets the axes. **(2 marks)**




(b) Solve $f(x) = x$ **(3 marks)**

$3x - 6 = x$
$3x = x + 6$
$2x = 6$
$x = 3$

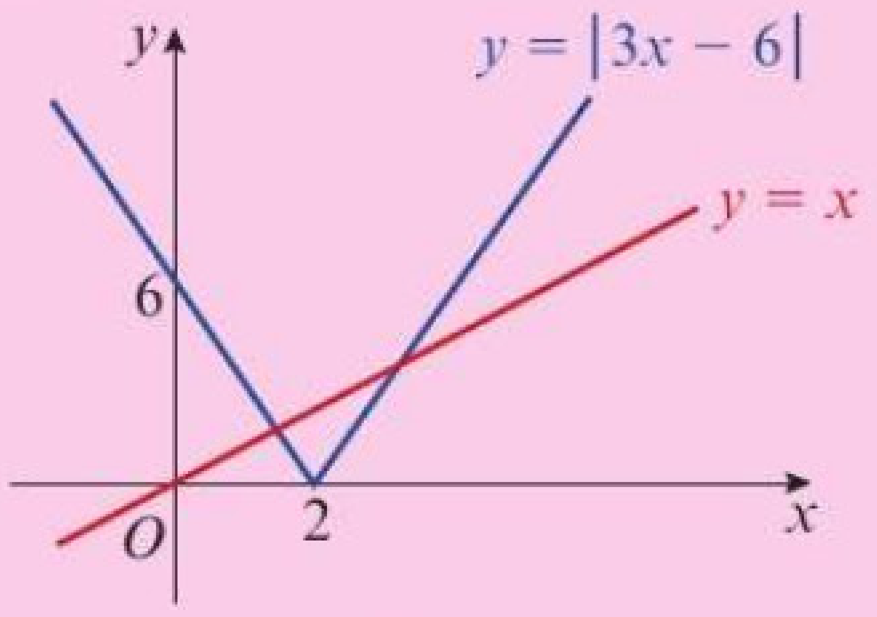
$-(3x - 6) = x$
$-3x + 6 = x$
$6 = 4x$
$x = \frac{3}{2}$

## Problem solved!

To solve $|3x - 6| = x$ you need to solve **two** equations:

- Positive argument: $3x - 6 = x$
- Negative argument: $-(3x - 6) = x$

Use your graph to check that the answers definitely exist:




If $y = x$ crosses $y = |3x - 6|$ twice then there are two solutions.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Here is a foolproof way of solving equations involving a modulus:

1. Rearrange the equation so the modulus is on one side.
2. Solve the equation with a positive argument.
3. Solve the equation with a negative argument.
4. Use a graph or plug the answers back into the original equation to check that they exist.

## Worked example

Solve $4 - |x + 2| = \frac{1}{2}x$ **(5 marks)**

$|x + 2| = 4 - \frac{1}{2}x$

$x + 2 = 4 - \frac{1}{2}x$
$\frac{3}{2}x = 2$
$x = \frac{4}{3}$

$-(x + 2) = 4 - \frac{1}{2}x$
$-x - 2 = 4 - \frac{1}{2}x$
$-6 = \frac{1}{2}x$
$x = -12$

Check: $4 - \left|\frac{4}{3} + 2\right| = 4 - \frac{10}{3} = \frac{2}{3} = \frac{1}{2}\left(\frac{4}{3}\right)$ ✓
$4 - |-12 + 2| = 4 - |-10|$
$= 4 - 10 = -6 = \frac{1}{2}(-12)$ ✓

## Now try this

1 The function f is defined by

$f: x \mapsto |2x + 4|, \quad x \in \mathbb{R}$

Answer (a) Sketch the graph with equation $y = f(x)$, showing the coordinates of the points where the graph cuts or meets the axes. **(2 marks)**

(b) Explain why the equation $f(x) = x$ has no solutions. Answer **(1 mark)**

(c) Solve $f(x) = -x$ Answer **(3 marks)**

2 Solve $2x + 1 = 5 - |x - 1|$ **(5 marks)**

Answer

Be careful. This equation has only **one** solution. Find separate solutions for the positive and negative arguments, then plug them both back into the equation to check which one is valid.

# Modulus transformations

You revised these transformations of the graph of $y = \mathrm{f}(x)$ on pages 13 and 14:

- $y = \mathrm{f}(x) + a$ Translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$
- $y = \mathrm{f}(x + a)$ Translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
- $y = a\mathrm{f}(x)$ Vertical stretch, scale factor $a$
- $y = \mathrm{f}(ax)$ Horizontal stretch, scale factor $\frac{1}{a}$
- $y = -\mathrm{f}(x)$ Reflection in the $x$-axis
- $y = \mathrm{f}(-x)$ Reflection in the $y$-axis.

You need to be able to combine these transformations and use the modulus function to sketch more complicated transformations.

## Golden rule

Carry out transformations in this order:

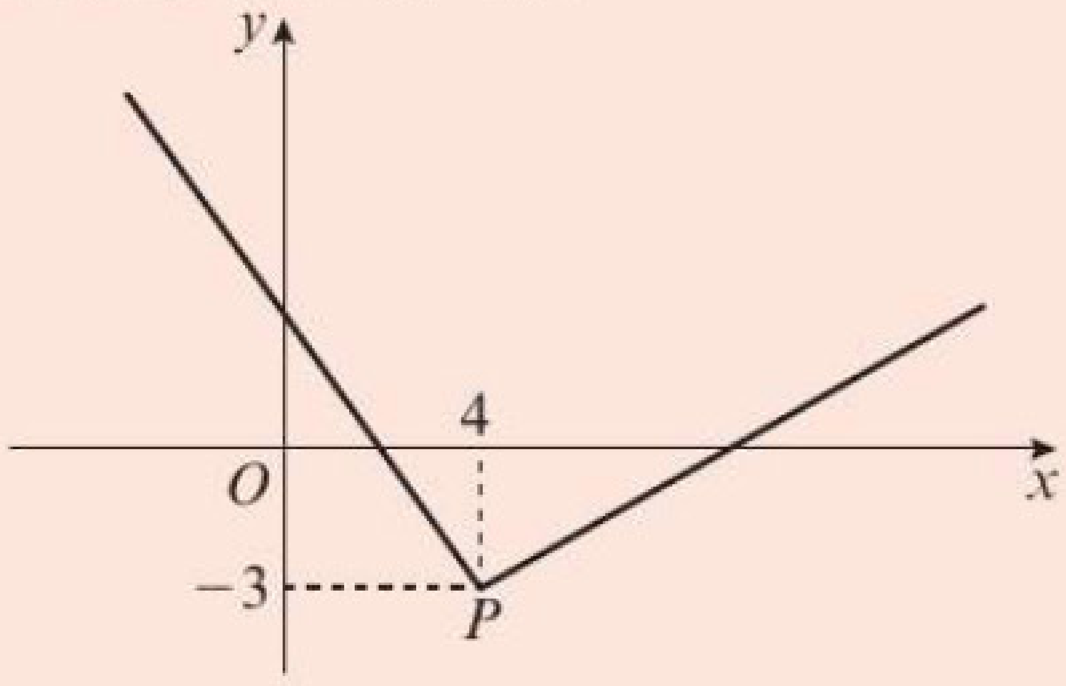
1. Anything 'inside' the function brackets
2. Multiples or modulus of the whole function
3. Addition or subtraction outside the function brackets.

$$y = \tfrac{1}{2}\mathrm{f}(|x|) + 4$$

(2) → $\tfrac{1}{2}$, (1) → $|x|$, (3) → $+ 4$

## Worked example

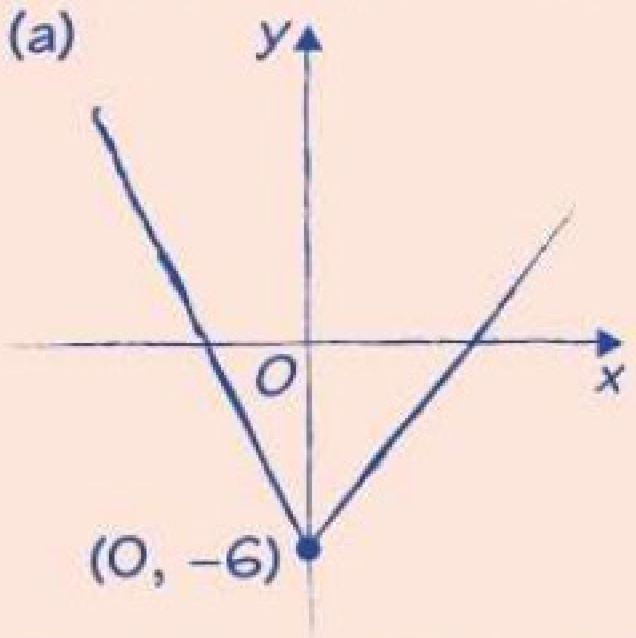
The diagram shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$. The graph consists of two line segments that meet at the point $P(4, -3)$.

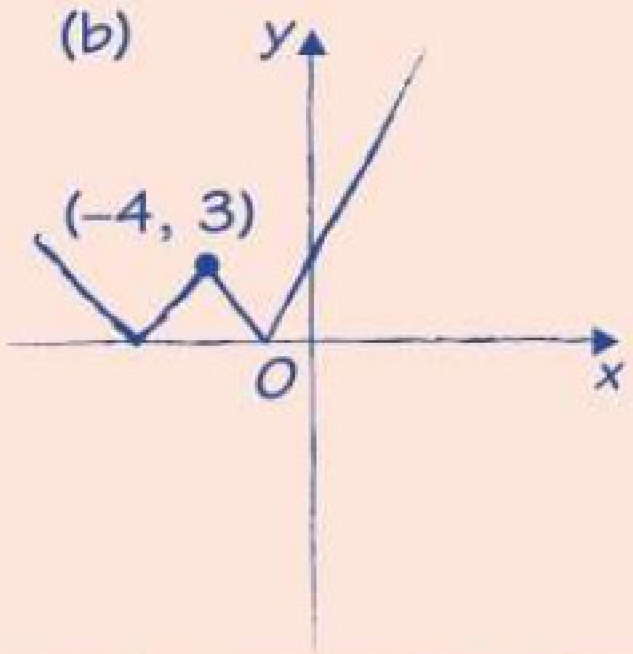



Sketch, on separate diagrams, the graphs of

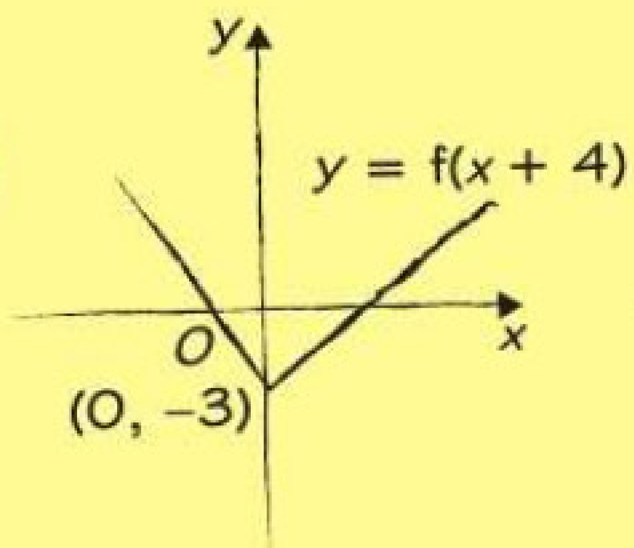
(a) $y = 2\mathrm{f}(x + 4)$ **(3 marks)** (b) $y = |\mathrm{f}(-x)|$ **(3 marks)**

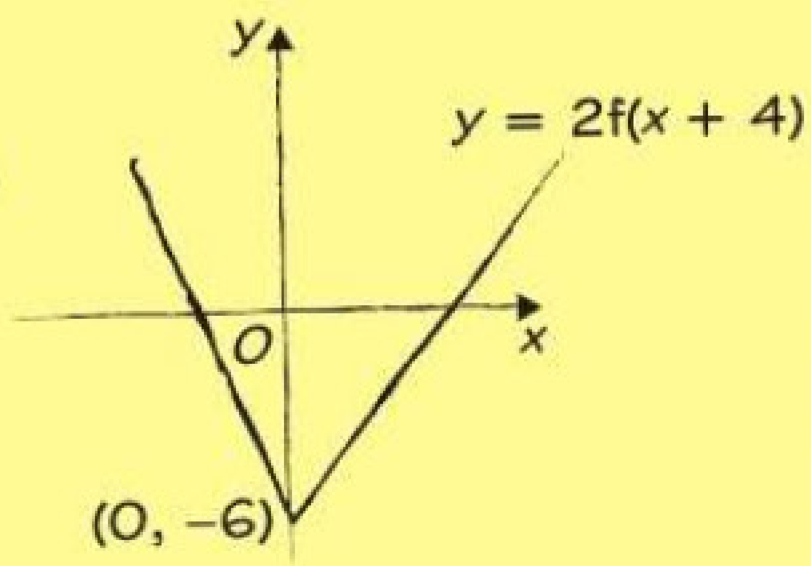
On each diagram, show the coordinates of the point corresponding to $P$.



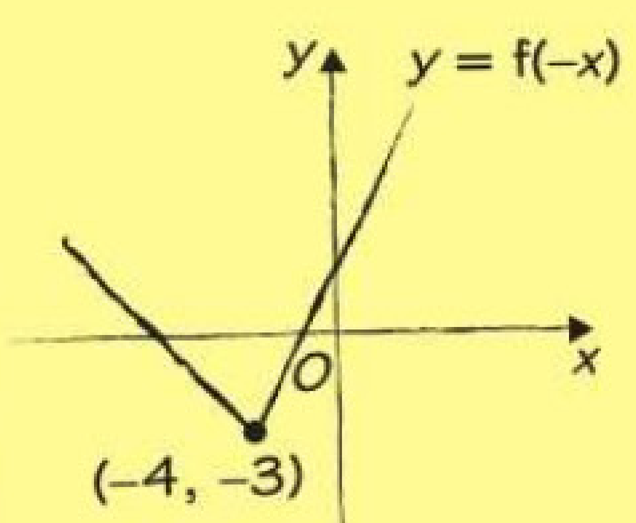



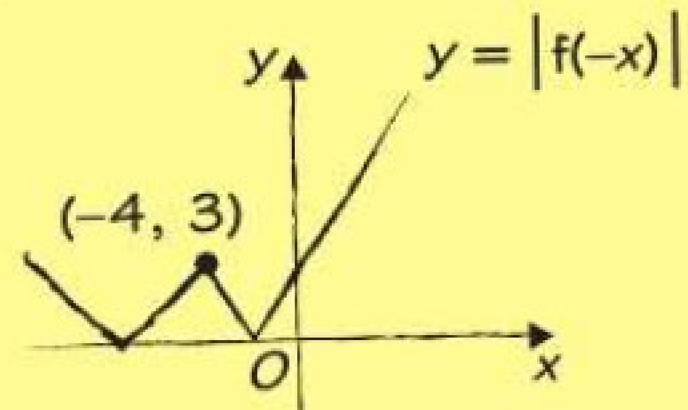

For part (a) you need to carry out a translation $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ followed by a vertical stretch with scale factor 2.




For part (b) you need to carry out a reflection in the $y$-axis followed by a modulus.

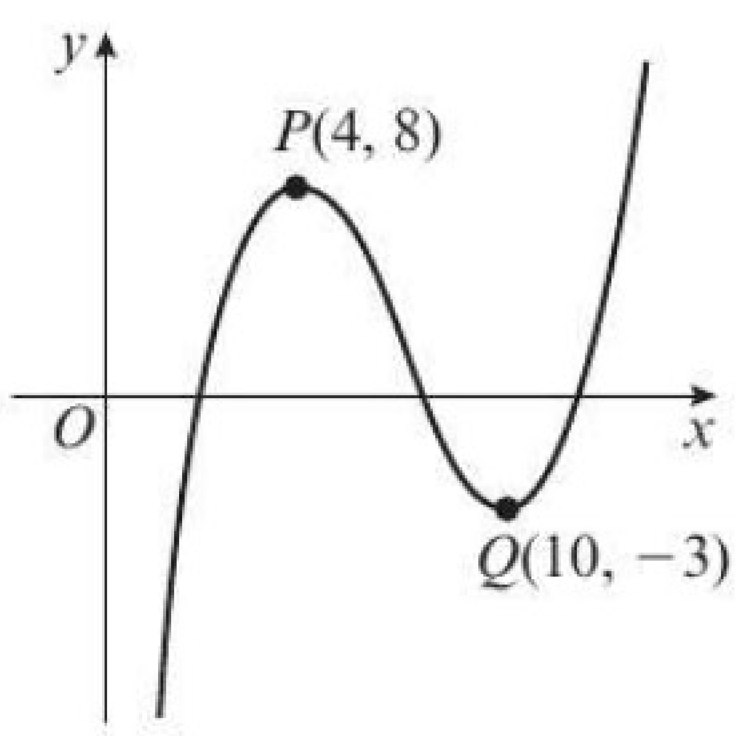



Have a look at page 64 for a reminder about sketching the modulus of a function.

## Now try this

The diagram shows a sketch of $y = \mathrm{f}(x)$. The graph has turning points at $P$ and $Q$.




(a) Write down the coordinates of the point to which $Q$ is transformed on the curve with equation

Answer (i) $y = 2\mathrm{f}(2x)$ (ii) $y = |\mathrm{f}(x + 4)|$ Answer **(4 marks)**

(b) Sketch, on separate diagrams, the graphs of

Answer (i) $y = \mathrm{f}(-x) + 3$ (ii) $y = -|\mathrm{f}(x)|$ Answer **(6 marks)**

Indicate on each diagram the coordinates of any turning points.