

## Summary of key points

- 1 This form of the binomial expansion can be applied to negative or fractional values of  $n$  to obtain an infinite series:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots + \frac{n(n-1)\dots(n-r+1)x^r}{r!} + \dots$$

The expansion is valid when  $|x| < 1$ ,  $n \in \mathbb{R}$ .

- 2 The expansion of  $(1 + bx)^n$ , where  $n$  is negative or a fraction, is valid for  $|bx| < 1$ , or  $|x| < \frac{1}{|b|}$ .
- 3 The expansion of  $(a + bx)^n$ , where  $n$  is negative or a fraction, is valid for  $\left|\frac{b}{a}x\right| < 1$  or  $|x| < \left|\frac{a}{b}\right|$ .



# Binomial expansion 2

You need to be able to use the binomial theorem to find a **series expansion** of expressions in the form  $(a + bx)^n$ , where  $n$  is **any real number**. You need to use this version of the binomial series, which is given in the formulae booklet:

The expansion is only valid for values of  $x$  between  $-1$  and  $1$ .

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times \dots \times r} x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

If you are given an expression in the form  $(a + bx)^n$  you will need to rearrange it by taking out a factor of  $a^n$  like this:  $(a + bx)^n = a^n \left(1 + \frac{bx}{a}\right)^n$ . In this case the expansion is valid for  $\left|\frac{bx}{a}\right| < 1$  or  $|x| < \frac{a}{b}$

## Worked example

$$f(x) = \frac{1}{\sqrt{4 + 5x}}$$

- (a) Find the binomial expansion of  $f(x)$ , in ascending powers of  $x$ , as far as the term in  $x^3$ , and state the range of values of  $x$  for which it is valid. **(6 marks)**

$$\begin{aligned} f(x) &= (4 + 5x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}} \left(1 + \frac{5x}{4}\right)^{-\frac{1}{2}} \\ &= \frac{1}{2} \left(1 + \frac{5x}{4}\right)^{-\frac{1}{2}} \\ &= \frac{1}{2} \left[ 1 + \left(-\frac{1}{2}\right)\left(\frac{5x}{4}\right) + \left(\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1 \times 2}\right)\left(\frac{5x}{4}\right)^2 + \left(\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \times 2 \times 3}\right)\left(\frac{5x}{4}\right)^3 + \dots \right] \\ &= \frac{1}{2} \left[ 1 - \frac{5}{8}x + \frac{75}{128}x^2 - \frac{625}{1024}x^3 + \dots \right] \\ &= \frac{1}{2} - \frac{5}{16}x + \frac{75}{256}x^2 - \frac{625}{2048}x^3 + \dots \quad \text{Valid for } |x| < \frac{4}{5} \end{aligned}$$

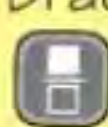
- (b) Hence find the coefficient of  $x$  in the series expansion of  $\frac{3 + x}{\sqrt{4 + 5x}}$  **(4 marks)**

$$\frac{3 + x}{\sqrt{4 + 5x}} = (3 + x) \left( \frac{1}{2} - \frac{5}{16}x + \frac{75}{256}x^2 - \frac{625}{2048}x^3 + \dots \right)$$

$$\begin{aligned} x \text{ term in expansion} &= (3) \left(-\frac{5}{16}x\right) + (x) \left(\frac{1}{2}\right) \\ &= \left(-\frac{15}{16} + \frac{1}{2}\right)x = -\frac{7}{16}x \end{aligned}$$

So coefficient of  $x$  is  $-\frac{7}{16}$

Start by writing the expression in the form  $(a + bx)^n$ , then take out a factor of  $a^n$ . You can now use the binomial series from the formulae booklet, with  $n = -\frac{1}{2}$ , replacing  $x$  with  $\frac{5x}{4}$

A common mistake is not to square or cube all of  $\frac{5x}{4}$ , so use brackets when you write out the series. You can work out the coefficients in one go on your calculator using brackets and the  key.

## Problem solved!

In your exam, you might be asked to find a binomial series in lots of ways:

- 'Find the binomial expansion of ...'
- 'Use the binomial theorem to expand ...'
- 'Find the series expansion of ...'
- 'Expand  $f(x)$  in ascending powers of  $x$  ...'

You will usually be told the highest power of  $x$  you need to find.

You will need to use problem-solving skills throughout your exam – **be prepared!**



## Now try this

- (a) Expand  $\sqrt[3]{1 - 6x}$ , in ascending powers of  $x$  up to and including the  $x^3$  term, simplifying each term. State the range of values of  $x$  for which the expansion is valid. **(4 marks)**
- (b) Use your expansion, with a suitable value of  $x$ , to obtain an approximation to  $\sqrt[3]{0.94}$ . Give your answer to 6 decimal places. **(2 marks)**

You must use your answer to part (a). To find a suitable value of  $x$ , solve  $1 - 6x = 0.94$