Summary of key points

1 This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots + \frac{n(n-1)\ldots(n-r+1)x^r}{r!} + \ldots$$

The expansion is valid when $|x| < 1$, $n \in \mathbb{R}$.

2 The expansion of $(1 + bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \frac{1}{|b|}$.

3 The expansion of $(a + bx)^n$, where n is negative or a fraction, is valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \left|\frac{a}{b}\right|$.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Binomial expansion 2

You need to be able to use the binomial theorem to find a **series expansion** of expressions in the form $(a + bx)^n$, where n is **any real number**. You need to use this version of the binomial series, which is given in the formulae booklet:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \times 2 \times \ldots \times r}x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

The expansion is only valid for values of x between -1 and 1.

If you are given an expression in the form $(a + bx)^n$ you will need to rearrange it by taking out a factor of a^n like this: $(a + bx)^n = a^n\left(1 + \frac{bx}{a}\right)^n$. In this case the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ or $|x| < \frac{a}{b}$

Worked example

$$f(x) = \frac{1}{\sqrt{4+5x}}$$

(a) Find the binomial expansion of $f(x)$, in ascending powers of x, as far as the term in x^3, and state the range of values of x for which it is valid. **(6 marks)**

$$f(x) = (4+5x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1 + \frac{5x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 + \frac{5x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{5x}{4}\right) + \left(\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1 \times 2}\right)\left(\frac{5x}{4}\right)^2 + \left(\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \times 2 \times 3}\right)\left(\frac{5x}{4}\right)^3 + \ldots\right]$$

$$= \frac{1}{2}\left[1 - \frac{5}{8}x + \frac{75}{128}x^2 - \frac{625}{1024}x^3 + \ldots\right]$$

$$= \frac{1}{2} - \frac{5}{16}x + \frac{75}{256}x^2 - \frac{625}{2048}x^3 + \ldots \qquad \text{Valid for } |x| < \frac{4}{5}$$

Start by writing the expression in the form $(a + bx)^n$, then take out a factor of a^n. You can now use the binomial series from the formulae booklet, with $n = -\frac{1}{2}$, replacing x with $\frac{5x}{4}$

A common mistake is not to square or cube **all** of $\frac{5x}{4}$, so use brackets when you write out the series. You can work out the coefficients in one go on your calculator using brackets and the key.

(b) Hence find the coefficient of x in the series expansion of $\frac{3+x}{\sqrt{4+5x}}$ **(4 marks)**

$$\frac{3+x}{\sqrt{4+5x}} = (3 + x)\left(\frac{1}{2} - \frac{5}{16}x + \frac{75}{256}x^2 - \frac{625}{2048}x^3 + \ldots\right)$$

$$x \text{ term in expansion} = (3)\left(-\frac{5}{16}x\right) + (x)\left(\frac{1}{2}\right)$$

$$= \left(-\frac{15}{16} + \frac{1}{2}\right)x = -\frac{7}{16}x$$

So coefficient of x is $-\frac{7}{16}$

Problem solved!

In your exam, you might be asked to find a binomial series in lots of ways:

- 'Find the binomial expansion of …'
- 'Use the binomial theorem to expand …'
- 'Find the series expansion of …'
- 'Expand f(x) in ascending powers of x …'

You will usually be told the highest power of x you need to find.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

(a) Expand $\sqrt[3]{1-6x}$, in ascending powers of x up to and including the x^3 term, simplifying each term. State the range of values of x for which the expansion is valid. **(4 marks)**

Answer

(b) Use your expansion, with a suitable value of x, to obtain an approximation to $\sqrt[3]{0.94}$. Give your answer to 6 decimal places. **(2 marks)**

Answer

You must use your answer to part (a). To find a suitable value of x, solve $1 - 6x = 0.94$