Summary of key points

4 A **geometric sequence** has a **common ratio** between consecutive terms.

5 The formula for the nth term of a geometric sequence is $\boldsymbol{u_n = ar^{n-1}}$, where a is the first term and r is the common ratio.

6 The sum of the first n terms of a geometric series is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1 \qquad \text{or} \qquad S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

where a is the first term and r is the common ratio.

7 A geometric series is convergent if and only if $|r| < 1$, where r is the common ratio.

The **sum to infinity** of a convergent geometric series is given by $S_\infty = \dfrac{a}{1 - r}$

Geometric sequences

To get from one term to the next in a geometric sequence you multiply by the same number each time. You usually use a to represent the **first term** and r to represent the **common ratio**. So the sequence is of the form $a, ar, ar^2, \dots, ar^{n-1}, \dots$ Here are two examples:

1 $\times 2 \quad \times 2 \quad \times 2 \quad \times 2$

$3, \ 6, \ 12, \ 24, \ 48 \dots$

$a = 3, r = 2$, nth term $= 3 \times 2^{n-1}$

2 $\times(-\frac{1}{2}) \quad \times(-\frac{1}{2}) \quad \times(-\frac{1}{2}) \quad \times(-\frac{1}{2})$

$80, \ -40, \ 20, \ -10, \ 5 \dots$

$a = 80, r = -\frac{1}{2}$, nth term $= 80 \times (-\frac{1}{2})^{n-1}$

General term

If a geometric sequence has first term a and common ratio r, then the nth term is

$$u_n = ar^{n-1}$$

You need to learn this formula – it's not given in the exam.

Worked example

The fifth term of a geometric sequence is 324 and the eighth term is 12

(a) Show that the common ratio is $\frac{1}{3}$ **(2 marks)**

$ar^4 = 324$ ①

$ar^7 = 12$ ②

② ÷ ①: $r^3 = \frac{1}{27}$

$r = \frac{1}{3}$

(b) Find the first term of the sequence. **(2 marks)**

$a\left(\frac{1}{3}\right)^4 = 324$

$\frac{1}{81}a = 324$

$a = 26244$

To get from the fifth term to the eighth term you multiply by r three times, so in total you have multiplied by r^3:

$\times r^3$

$\times r \quad \times r \quad \times r$

$\dots, \ 324, \ ?, \ ?, \ 12,$

Use the formula for the nth term to write two equations. Divide one equation by the other to eliminate a, then take the cube root of both sides to work out r. You can substitute your value of r into one of your equations to find the value of a.

In a **geometric** sequence the **ratio** between consecutive terms is constant. This gives you this useful relationship:

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \dots$$

Substituting $u_1 = 9$, $u_2 = k$ and $u_3 = (k + 4)$ into this relationship gives you a quadratic equation, which you can solve to find two possible values of k.

Worked example

The first three terms of a geometric sequence are 9, k and $(k + 4)$ respectively.

Find the two possible values of k. **(5 marks)**

$\frac{k}{9} = \frac{k+4}{k}$

$k^2 = 9(k + 4)$

$k^2 - 9k - 36 = 0$

$(k - 12)(k + 3) = 0$

$k = 12$ or $k = -3$

Now try this

1 A geometric sequence has first term $a = 420$ and common ratio $r = \frac{5}{6}$. Answer
Find the 20th term of the sequence. Give your answer to 3 significant figures. **(2 marks)**

In part (c), the common ratio will be $\frac{k}{k-6}$

2 The first three terms of a geometric sequence are $(k - 6)$, k and $(2k + 5)$ respectively, where k is a positive constant.

(a) Show that $k^2 - 7k - 30 = 0$ **(4 marks)** Answer

(b) Hence show that $k = 10$ **(2 marks)** Answer

(c) Write down the common ratio for this sequence. **(1 mark)** Answer

Geometric series

In a series, the terms are **added together**. The terms in a geometric series have a **common ratio**. You write *a* for the **first term** and *r* for the common ratio. Here are two examples.

1 ×3 ×3 ×3 ×3

$2 + 6 + 18 + 54 + 162 + \ldots$

$a = 2$ and $r = 3$

The sum of the first 5 terms is $S_5 = 242$

2 $\times(-\frac{1}{2})$ $\times(-\frac{1}{2})$ $\times(-\frac{1}{2})$

$30 - 15 + 7.5 - 3.75 + \ldots$

$a = 30$ and $r = -\frac{1}{2}$. When *r* is negative, the terms **alternate** between + and −. You can still use the formula to work out S_n:

$$S_4 = \frac{30\left[1 - \left(-\frac{1}{2}\right)^4\right]}{1 - \left(-\frac{1}{2}\right)} = 18.75$$

Sum to *n* terms

If a **geometric** series has first term *a* and common ratio *r*, then the **sum** of the first *n* terms is

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r - 1}$$

You can use either of these versions. The first one appears in the formulae booklet. You also need to know its **proof** and this is **not** in the formulae booklet. Look at the Worked example below to see the proof in action.

Problem solved!

To prove this you need to write out S_n and then **multiply every term by** *r*. If you subtract rS_n from S_n most of the terms cancel:

$a + \cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$

$- \cancel{ar} - \cancel{ar^2} - \ldots - \cancel{ar^{n-2}} - \cancel{ar^{n-1}} - ar^n$

You are left with $S_n - rS_n = a - ar^n$. You can factorise the left-hand side and divide by $(1 - r)$ to get S_n on its own.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

A geometric series has first term *a* and common ratio *r*. Prove that the sum of the first *n* terms of the series, S_n, is $\frac{a(1 - r^n)}{1 - r}$ **(4 marks)**

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ ①

$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$ ②

① − ②: $S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

$S_n = \frac{a(1 - r^n)}{1 - r}$

Worked example

Find $\sum_{k=1}^{10} 50(2^k)$ **(3 marks)**

$\sum_{k=1}^{10} 50(2^k) = 100 + 200 + 400 + \ldots + 51\,200$

So $a = 100$ and $r = 2$

$S_{10} = \frac{100(1 - 2^{10})}{1 - 2} = 102\,300$

For a reminder about **sigma notation** have a look at page 72. It's a good idea to write out a few terms to check that you have the correct values for *a* and *r*. You can enter the calculation in one go on your calculator using the [fraction] and [$x^{\blacksquare}$] keys.

Now try this

You can't use the formula. Try to apply the technique in the first Worked example.

1 A geometric series has first term $a = 7$ and common difference $r = \frac{3}{2}$. Find the sum of the first 20 terms of the series, giving your answer to the nearest whole number. **(2 marks)**

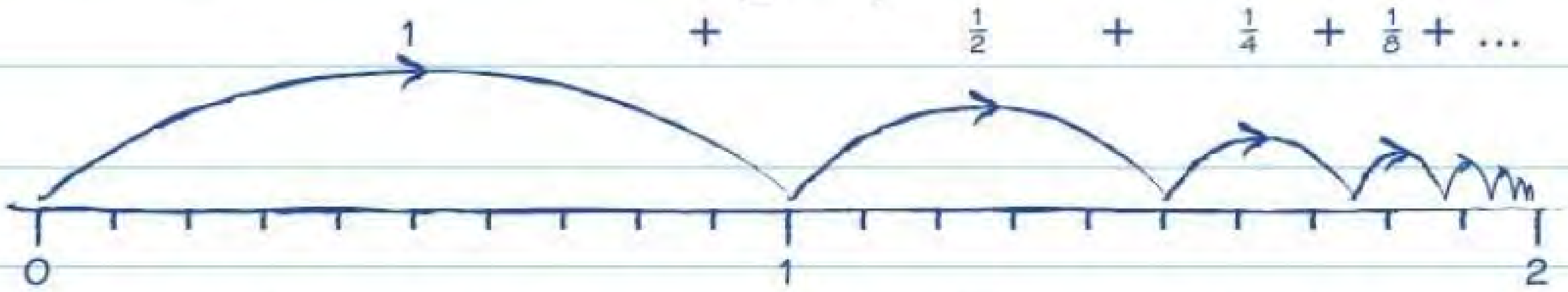
Answer

2 In the geometric series $1 + 2 + 4 + 8 + \ldots$ each term is twice the previous term. Prove that the sum of the first *n* terms of this series is $2^n - 1$ **(4 marks)**

Answer

Infinite series

This diagram shows the **geometric series** $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ with first term 1 and common ratio $\frac{1}{2}$



The sum gets closer and closer to 2, but never reaches it. You say that the **sum of infinity** of this series is 2. You write the sum to infinity as S_∞.

Sum to infinity

For a geometric series with first term a and common difference r, the sum to infinity is

$S_\infty = \frac{a}{1-r}$ where $-1 < r < 1$

In the example above, $a = 1$ and $r = \frac{1}{2}$, so $S_\infty = \frac{1}{1-\frac{1}{2}} = 2$

Convergent series

The sum to infinity of a geometric series **only exists** if the series is **convergent**. This only happens if the common ratio, r, is between –1 and 1.

In the formulae booklet this condition is written as $|r| < 1$.

Be careful – the inequalities are strict. Geometric series with $r > 1$, $r = 1$, $r < -1$ or $r = -1$ **are not convergent**, so S_∞ doesn't exist.

In the example above, $r = \frac{1}{2}$ so the series is convergent.

Worked example

Find the sum to infinity of the geometric series

$\frac{3}{5} + \frac{6}{15} + \frac{12}{45} + \ldots$ **(3 marks)**

$a = \frac{3}{5}$, $r = \frac{6}{15} \div \frac{3}{5} = \frac{2}{3}$

$S_\infty = \frac{\frac{3}{5}}{1 - \frac{2}{3}} = \frac{9}{5}$

You need to find the values of a and r before you can use the formula for S_∞. a is the first term and r is the common ratio, so $r = \frac{u_2}{u_1} = \frac{6}{15} \div \frac{3}{5}$. You can check your value of r using u_2 and u_3: $\frac{u_3}{u_2} = \frac{12}{45} \div \frac{6}{15} = \frac{2}{3}$ ✓

Worked example

The first term of a geometric series is 80 and the sum to infinity is 50. Find the common ratio, r. **(3 marks)**

$\frac{80}{1-r} = 50$

$80 = 50(1 - r)$

$50r = -30$

$r = -\frac{3}{5}$

Use the formula for S_∞ to write an equation and solve it to find r. Remember that r can be positive as well as negative. This is the convergent geometric series

$80 - 48 + \frac{144}{5} - \frac{432}{25} + \ldots$

which has sum to infinity $\frac{80}{1-\left(-\frac{3}{5}\right)} = 50$

Now try this

1 Find the sum to infinity of the geometric series

$15 + 12 + 9.6 + \ldots$ Answer **(3 marks)**

2 The first term of a geometric series is 2 and the common ratio is k. The sum to infinity of the series is $3k + 4$, where k is a constant. Find the value of k. **(5 marks)**

Answer

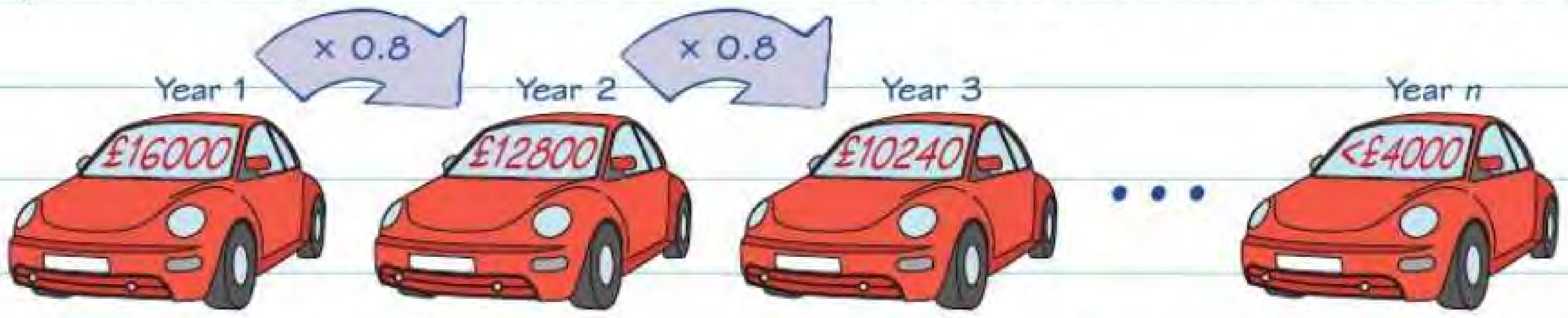
Use the formula for S_∞ to write a quadratic equation involving k. Remember that in order for the sum to infinity to exist, k must be between –1 and 1.

Series and logs

You can use **logarithms** to answer questions involving **geometric sequences** and **series**. For a reminder about working with exponentials and logs, look at page 50.

Modelling with geometric sequences

A car costs £16 000 new and depreciates in value by 20% each year. Its value each year produces a geometric sequence with first term $a = 16\,000$, and common ratio $r = 0.8$, or $\frac{4}{5}$



You can use logs to work out how many years it takes until the car is worth less than £4000.

After n years the car is worth $16\,000 \times (0.8)^{n-1}$. You need this value to be less than £4000, so

$$16\,000 \times (0.8)^{n-1} < 4000$$

$$(0.8)^{n-1} < 0.25$$

Don't try to write $\log_{0.8} 0.25 < n - 1$ and use the $\log_{\square}\square$ key on your calculator – it's safer to **take logs** of both sides.

$$\log (0.8)^{n-1} < \log 0.25$$

$$(n-1)\log 0.8 < \log 0.25$$

$$n - 1 > \frac{\log 0.25}{\log 0.8}$$

$\log 0.8$ is **negative** so when you divide both sides by $\log 0.8$ you have to **reverse** the inequality.

$$n > 7.212...$$

n must be a whole number, so the answer is $n = 8$

Worked example

A geometric series has first term 5 and common ratio $\frac{3}{2}$. Find the smallest value of n for which the sum of the first n terms of the series exceeds 12 000 **(4 marks)**

$a = 5$, $r = 1.5$, $S_n = \frac{a(1 - r^n)}{1 - r}$

$$\frac{5(1 - 1.5^n)}{1 - 1.5} > 12\,000$$

$$5(1 - 1.5^n) < -6000$$

$$1 - 1.5^n < -1200$$

$$1.5^n > 1201$$

$$\log(1.5^n) > \log 1201$$

$$n \log 1.5 > \log 1201$$

$$n > \frac{\log 1201}{\log 1.5} = 17.488$$

$$n = 18$$

Problem solved!

Be really careful with **inequalities**.

- If you multiply or divide by a negative number, change the direction of the inequality sign.
- If $a < 1$ then $\log a$ is **negative**.
- Check that your answer makes sense. You're usually looking for the next integer **greater than** the value you calculate.
- Double-check whether you should be using the formula for the ***n*th term**, ar^{n-1}, or the formula for the **sum to *n* terms**, $\frac{a(1 - r^n)}{1 - r}$

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

1 A biologist models the number of trout after n years as $200 \times (1.4)^{n-1}$. Her model predicts that after k years, the number of trout in the lake will exceed 600.

(a) Show that $k > \frac{\log 3}{\log 1.4} + 1$ Answer **(2 marks)**

(b) Find the smallest possible value of k. **(2 marks)** Answer

2 A geometric series has first term 4 and common ratio $\frac{9}{10}$. Find the smallest value of n for which the sum of the first n terms of the series exceeds 30. **(4 marks)** Answer