Summary of key points

6 Sometimes you can simplify an integral by changing the variable. This process is similar to using the chain rule in differentiation and is called **integration by substitution**.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Integration by substitution

You can use a substitution to turn a complicated integral into a simpler one. If you have to use a substitution in your exam it will usually be given with the question.

Follow these steps to find $\int \frac{e^{3x}}{(e^{3x}+1)^2}\,dx$ using the substitution $u = e^{3x}$:

1 Find $\frac{du}{dx}$ and write an expression for dx in the form $f(x)\,du$

$u = e^{3x}$ so $\frac{du}{dx} = 3e^{3x}$

dx is **equivalent** to $\left(\frac{1}{3e^{3x}}\right)du$

This is an expression for dx in the form $f(x)\,du$.

2 Swap dx for $f(x)\,du$ in the integral and simplify if possible.

$$\int \frac{e^{3x}}{(e^{3x}+1)^2}\,dx = \int \frac{e^{3x}}{(e^{3x}+1)^2}\left(\frac{1}{3e^{3x}}\right)du = \int \frac{1}{3(e^{3x}+1)^2}\,du$$

3 Substitute every x to get an integral involving only u and du.

$$= \int \frac{1}{3(u+1)^2}\,du = \int \tfrac{1}{3}(u+1)^{-2}\,du$$

The new integral should be easier to find.

4 Integrate with respect to u.

$$= -\tfrac{1}{3}(u+1)^{-1} + c$$

5 Use your substitution in reverse to get an answer in terms of x.

$$= -\tfrac{1}{3}(e^{3x}+1)^{-1} + c$$

Worked example

Use the substitution $x = \sin\theta$ to find the exact value of $\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$ **(7 marks)**

$x = \sin\theta$ so $\frac{dx}{d\theta} = \cos\theta$, so $dx = \cos\theta\,d\theta$

When $x = \frac{1}{2}$, $\theta = \arcsin\frac{1}{2} = \frac{\pi}{6}$

When $x = 0$, $\theta = \arcsin 0 = 0$

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{(\cos^2\theta)^{\frac{3}{2}}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sec^2\theta\,d\theta$$

$$= \left[\tan\theta\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{3}$$

Transforming limits

When you use substitution for a **definite integral** you need to use the substitution to **transform** your limits.

Transform limits from values of x to values of θ.

$$\int_{x=0}^{x=\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx \rightarrow \int_{\theta=0}^{\theta=\frac{\pi}{6}} \sec^2\theta\,d\theta$$

You can now use your values of θ with the integrated expression to evaluate the integral.

You might need to use trigonometric identities in an integration by substitution. You can use $\sin^2\theta + \cos^2\theta \equiv 1$ to write the whole integral in terms of $\cos\theta$.

$$\frac{1}{(\cos^2\theta)^{\frac{3}{2}}}\cos\theta \equiv \frac{1}{\cos^3\theta}\cos\theta \equiv \frac{1}{\cos^2\theta}$$

Now try this

1 Use the substitution $u = 3 + \sin x$ to show that

$$\int \frac{\sin 2x}{(3+\sin x)^2}\,dx = 2\ln(3+\sin x) + \frac{6}{3+\sin x} + c$$

where c is a constant. Answer **(5 marks)**

2 Use the substitution $u^2 = 2x + 1$ to find the exact value of

$$\int_0^4 \frac{4x}{\sqrt{2x+1}}\,dx$$

(7 marks) Answer

You will need to use implicit differentiation to find the relationship between du and dx. Have a look at page 94 for a reminder.