Summary of key points

8 Partial fractions can be used to integrate algebraic fractions.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Integrating partial fractions

You can integrate some functions by writing them as **partial fractions**. You can revise this technique on page 58.

Once it is written in partial fractions, you can integrate the expression **term-by-term**.

$$\frac{4x^3+5}{x(2x-1)^2} = 1 + \frac{5}{x} + \frac{-8}{2x-1} + \frac{11}{(2x-1)^2}$$

This integrates to $5\ln|x|$

This integrates to $-4\ln|2x-1|$

Write this as $11(2x-1)^{-2}$ to integrate it. The result is $-\frac{11}{2}(2x-1)^{-1}$

$$\text{So } \int \frac{4x^3+5}{x(2x-1)^2}\,dx = x + 5\ln|x| - 4\ln|2x-1| - \frac{11}{2}(2x-1)^{-1} + c$$

Worked example

(a) Express $\frac{5x+3}{(2x-3)(x+2)}$ in partial fractions. **(3 marks)**

$$\frac{5x+3}{(2x-3)(x+2)} = \frac{A}{2x-3} + \frac{B}{x+2}$$

$5x + 3 = A(x+2) + B(2x-3)$

Let $x = \frac{3}{2}$: $5(\frac{3}{2}) + 3 = A(\frac{3}{2} + 2)$

$\underline{A = 3}$

Let $x = -2$: $5(-2) + 3 = B(2(-2) - 3)$

$\underline{B = 1}$

So $\frac{5x+3}{(2x-3)(x+2)} = \frac{3}{2x-3} + \frac{1}{x+2}$

(b) Hence find the exact value of $\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx$, giving your answer as a single logarithm. **(5 marks)**

$$\int_2^6 \left(\frac{3}{2x-3} + \frac{1}{x+2}\right)dx = \left[\frac{3}{2}\ln|2x-3| + \ln|x+2|\right]_2^6$$

$= \left(\frac{3}{2}\ln 9 + \ln 8\right) - \left(\frac{3}{2}\ln 1 + \ln 4\right)$

$= \ln 27 + \ln 8 - \ln 1 - \ln 4$

$= \ln 54$

The question says 'hence' so you need to use your partial fractions from part (a) to work out the integration. If you are doing **definite integration**, make sure you write out the integral before doing any substitutions. You can get method marks even if you make a mistake in your working.

Problem solved!

Use the **laws of logs** to simplify your answer:

$\frac{3}{2}\ln 9 = \ln 9^{\frac{3}{2}} = \ln(\sqrt{9})^3 = \ln 27$

Remember that $\ln a + \ln b = \ln ab$ and $\ln a - \ln b = \ln\frac{a}{b}$ so:

$\ln 27 + \ln 8 - \ln 1 - \ln 4 = \ln\left(\frac{27 \times 8}{1 \times 4}\right)$

$= \ln 54$

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

$$f(x) = \frac{18x^2+10}{9x^2-1} = A + \frac{B}{3x+1} + \frac{C}{3x-1}$$

(a) Find the values of the constants A, B and C. Answer **(4 marks)**

(b) Hence find $\int f(x)\,dx$ Answer **(3 marks)**

(c) Find $\int_1^2 f(x)\,dx$, giving your answer in the form $2 + \ln k$ where k is a constant to be found. Answer **(3 marks)**

$9x^2 - 1$ is a difference of two squares. Factorise it using $(a^2 - b^2) = (a+b)(a-b)$.

Use the laws of logs to simplify your expression for part (c). Remember that k doesn't have to be an integer.