Summary of key points

11 When $\frac{dy}{dx} = f(x)g(y)$ you can write

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Solving differential equations

You revised how to **form** differential equations on page 97. You can use your integration skills to **solve** differential equations:

1. **Separate** the variables onto different sides.
2. **Integrate** both sides. Only include **one** constant of integration.
3. **Substitute** the boundary conditions to find the value of c.
4. **Rearrange** the solution into the form needed.

Worked example

Water is being heated in a kettle. At time t seconds, the temperature of the water is θ °C. The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{d\theta}{dt} = \lambda(120 - \theta), \qquad \theta \leqslant 100$$

where λ is a positive constant.
Given that $\theta = 20$ when $t = 0$,

(a) solve this differential equation to show that $\theta = 120 - 100e^{-\lambda t}$ **(8 marks)**

$$\int \frac{1}{120 - \theta} d\theta = \int \lambda \, dt$$

$-\ln(120 - \theta) = \lambda t + c$

When $t = 0$, $\theta = 20$:

$-\ln(120 - 20) = \lambda(0) + c$

$c = -\ln 100$

So $-\ln(120 - \theta) = \lambda t - \ln 100$

$\ln(120 - \theta) = -\lambda t + \ln 100$

$120 - \theta = e^{-\lambda t + \ln 100}$

$= 100e^{-\lambda t}$

$\theta = 120 - 100e^{-\lambda t}$

When the temperature of the water reaches 100 °C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off. **(3 marks)**

$100 = 120 - 100e^{-0.01t}$

$e^{-0.01t} = 0.2$

$-0.01t = \ln 0.2 = -1.6094...$

$t = 161$ (nearest second)

Separating variables

You can't integrate expressions containing a mixture of different variables. Before you integrate both sides of a differential equation you need to separate the variables.

If $\frac{dy}{dx} = f(x)g(y)$ then $\int \frac{1}{g(y)} dy = \int f(x)\, dx$

For example:
$\frac{dy}{dx} = 2x^2(1 - 5y)^3 \Rightarrow \int \frac{1}{(1 - 5y)^3} dy = \int 2x^2 \, dx$

Remember that λ is a constant. You can keep it on either side of the equation when you separate variables. It's easier to leave it on the 'dt' side then integrate to get $\lambda t + c$.

Once you have integrated you have found a **general solution** to the differential equation. You could rearrange it at this point:

$-\ln(120 - \theta) = \lambda t + c$

$120 - \theta = e^{-\lambda t - c} = Ae^{-\lambda t}$

$\theta = 120 - Ae^{-\lambda t}$

A is a constant equal to e^{-c}. You could substitute the boundary conditions at this stage to find the **particular solution** asked for in the question.

Now try this

Separate the variables to get $\int \frac{1}{x} dx = \int \cos 2t \, dt$

1 (a) Find a general solution to the differential equation $(\sec 2t)\frac{dx}{dt} = x$ **(5 marks)**

Answer

(b) Find a particular solution to this equation given that $x = 2$ when $t = \frac{\pi}{4}$ **(2 marks)**

Answer

2 (a) Find $\int (2y + 1)^{-3} \, dy$ Answer **(2 marks)**

(b) Given that $y = 0.5$ at $x = -8$, solve the differential equation $\frac{dy}{dx} = \frac{(2y + 1)^3}{x^2}$, giving your answer in the form $y = f(x)$.

Answer **(6 marks)**