Summary of key points

2 To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

Iteration

An iterative process is one where an answer is fed back in as a new starting value. You can use an **iterative formula** to find numerical answers to a given degree of accuracy.

Worked example

$f(x) = x^3 + 2x^2 - 3x - 11$

(a) Show that $f(x) = 0$ can be rearranged as

$$x = \sqrt{\frac{3x+11}{x+2}}, \quad x \neq -2$$ **(2 marks)**

$x^3 + 2x^2 - 3x - 11 = 0$

$x^2(x + 2) - 3x - 11 = 0$

$x^2(x + 2) = 3x + 11$

$x^2 = \frac{3x+11}{x+2}$ so $x = \sqrt{\frac{3x+11}{x+2}}$

The equation $f(x) = 0$ has one positive root, α.

The iterative formula $x_{n+1} = \sqrt{\frac{3x_n+11}{x_n+2}}$ is used to find an approximation to α.

(b) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3 and x_4. **(3 marks)**

$x_2 = \sqrt{\frac{3(0)+11}{(0)+2}} = 2.34520788... = 2.345$ (3 d.p.)

$x_3 = \sqrt{\frac{3x_2+11}{x_2+2}} = 2.03732494... = 2.037$ (3 d.p.)

$x_4 = \sqrt{\frac{3x_3+11}{x_3+2}} = 2.05874811... = 2.059$ (3 d.p.)

(c) Show that $\alpha = 2.057$ correct to 3 decimal places. **(3 marks)**

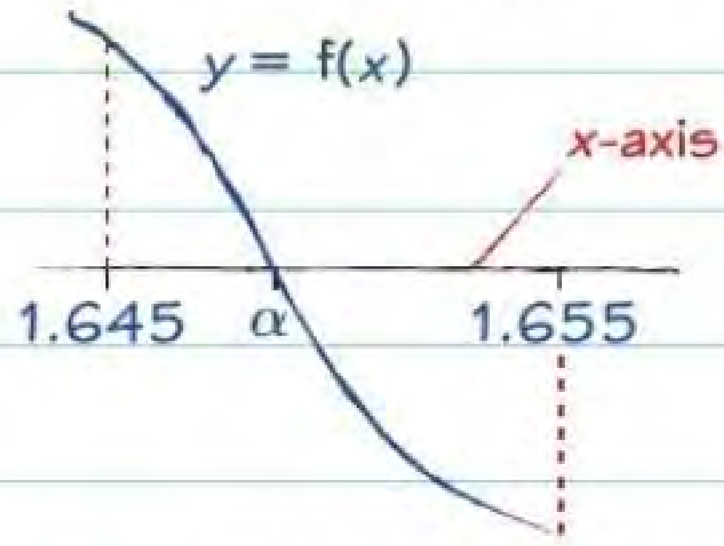
$f(2.0565) = (2.0565)^3 + 2(2.0565)^2 - 3(2.0565) - 11$
$= -0.01378...$ Negative

$f(2.0575) = (2.0575)^3 + 2(2.0575)^2 - 3(2.0575) - 11$
$= 0.0041401...$ Positive

Change of sign so $2.0565 < \alpha < 2.0575$, so $\alpha = 2.057$ correct to 3 d.p.

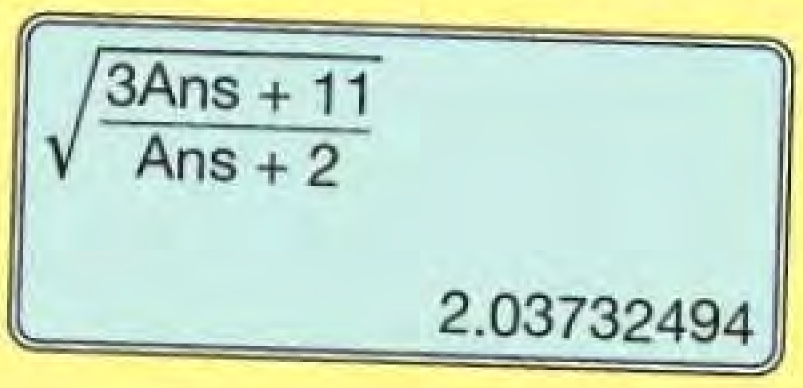
Change of sign

You can use a change of sign (from **positive** to **negative**, or vice versa) to show that a particular interval contains a root of an equation.

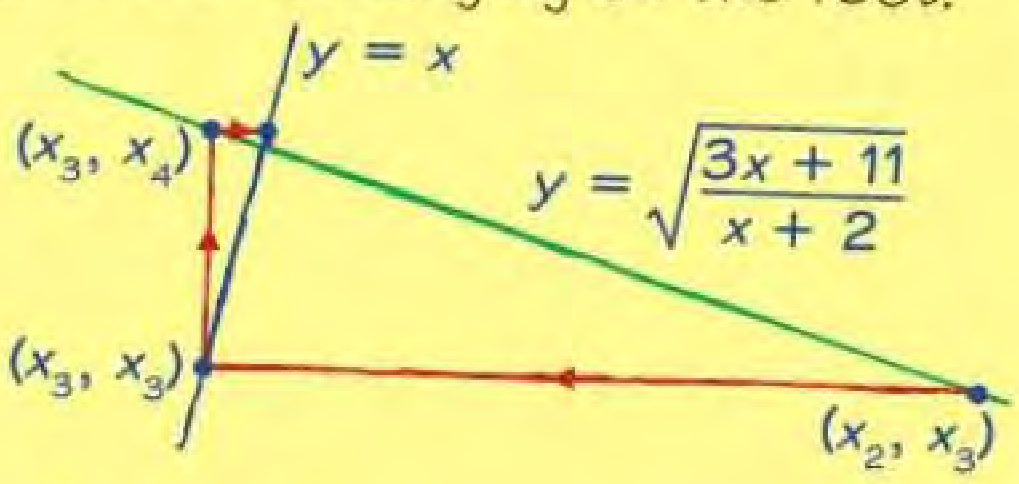


f(1.645) is positive and f(1.655) is negative so f(x) = 0 has a root, α, between 1.645 and 1.655. All values in this interval round to 1.65, so α = 1.65 to 2 decimal places.

You can iterate quickly using the **Ans** function on your calculator:



You can visualise this iteration using a **cobweb diagram**. This shows the x_n values converging on the root.



Now try this

$f(x) = \ln(x + 1) - 2x + 2, \quad x > 0$

The equation $f(x) = 0$ has one root, α.

(a) Show that $1 < \alpha < 2$. **(2 marks)**

Answer

(b) Use the iterative formula

Answer $x_{n+1} = \frac{1}{2}\ln(x_n + 1) + 1, \quad x_0 = 1.5$

to calculate values of x_1, x_2 and x_3, giving your answers to 5 decimal places. **(3 marks)**

(c) Show that $\alpha = 1.4475$ correct to 4 decimal places. **(3 marks)**

Answer