Summary of key points

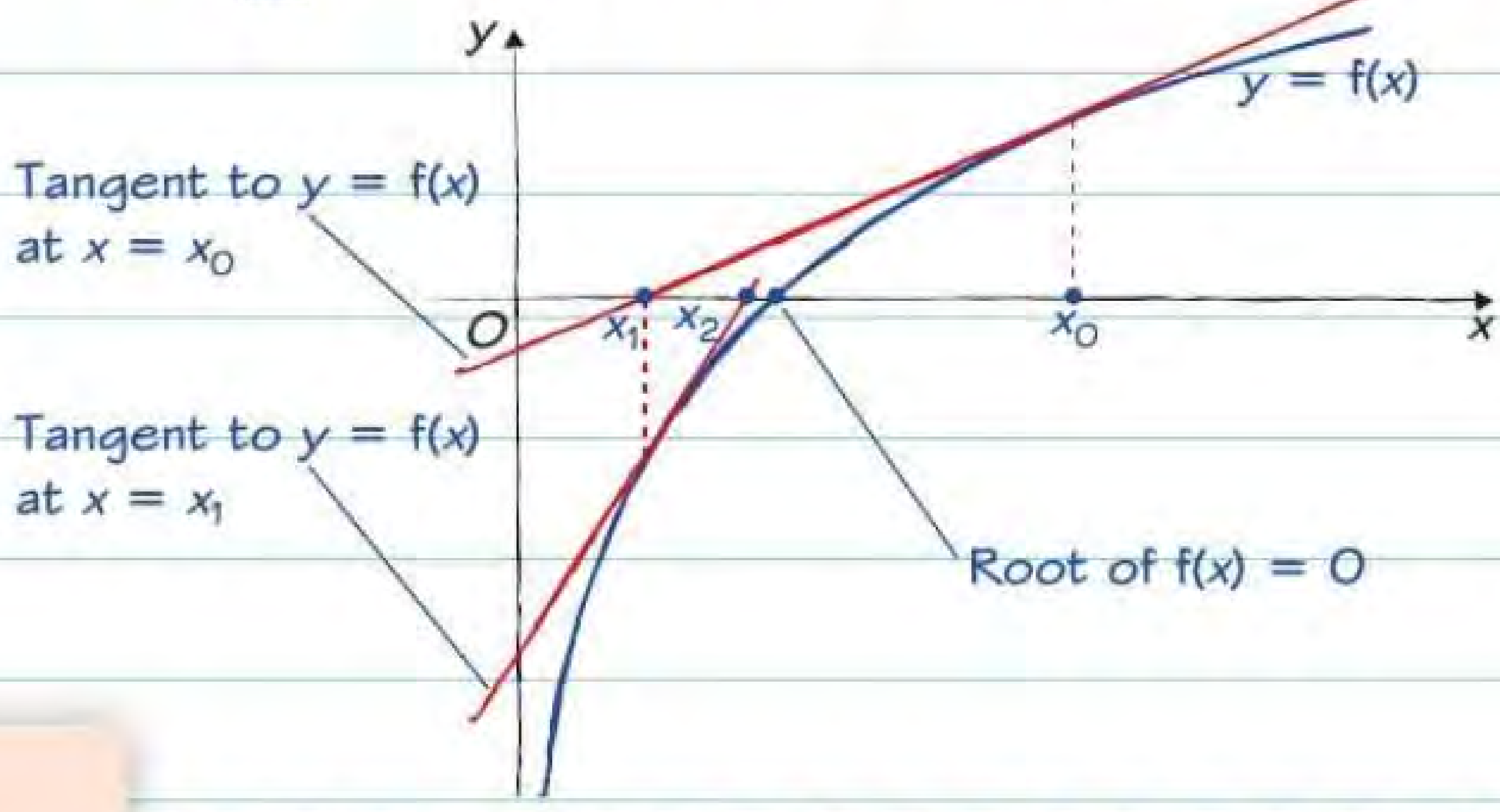
3 The Newton–Raphson formula for approximating the roots of a function f(x) is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Had a look ☐ Nearly there ☐ Nailed it! ☐

The Newton–Raphson method

You can use the Newton–Raphson method to find a numerical solution to an equation of the form f(x) = 0. The method works by using tangents to get closer and closer to a root. Each improved approximation, x_{n+1}, is the point where the tangent to the curve y = f(x) at $x = x_n$ crosses the x-axis.



Worked example

f(x) = $x^4 - e^x$

The equation f(x) = 0 has a root, α, in the interval [1, 2]. Taking $x_0 = 1.5$ as your starting value, apply the Newton-Raphson process once to find a second approximation to α.

(5 marks)

$f'(x) = 4x^3 - e^x$

With $x_0 = 1.5$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{1.5^4 - e^{1.5}}{4 \times 1.5^3 - e^{1.5}}$$

$$= 1.436 \text{ (3 d.p.)}$$

Golden rule

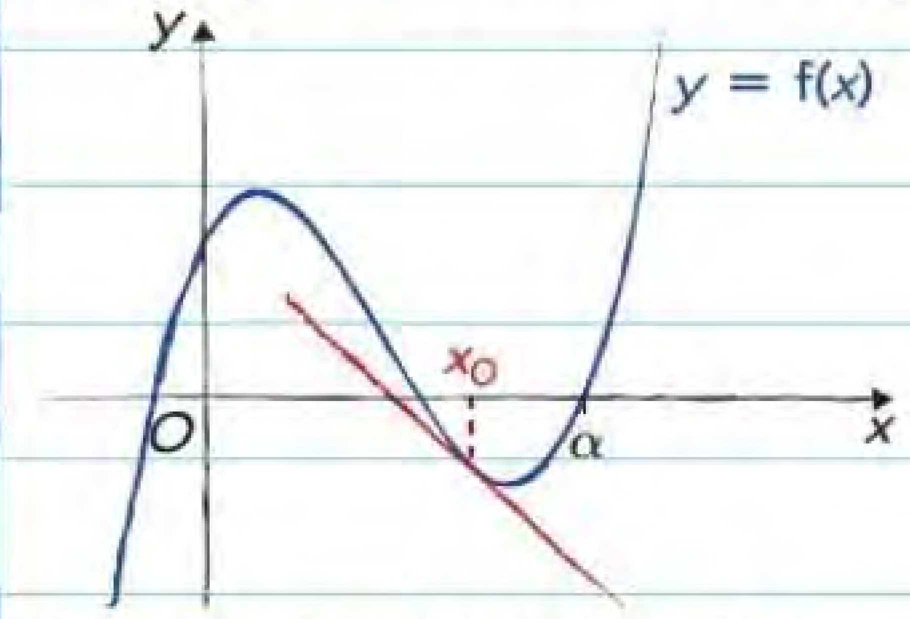
The formula for the Newton–Raphson method is given in the formulae booklet:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

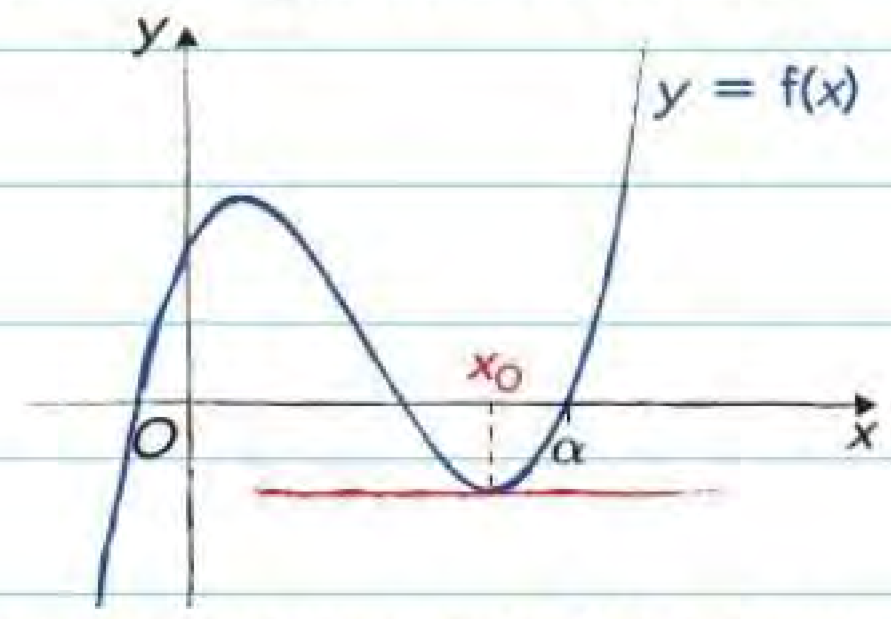
Start by differentiating f(x). Make sure you check that your final answer is in the correct interval.

Failure cases

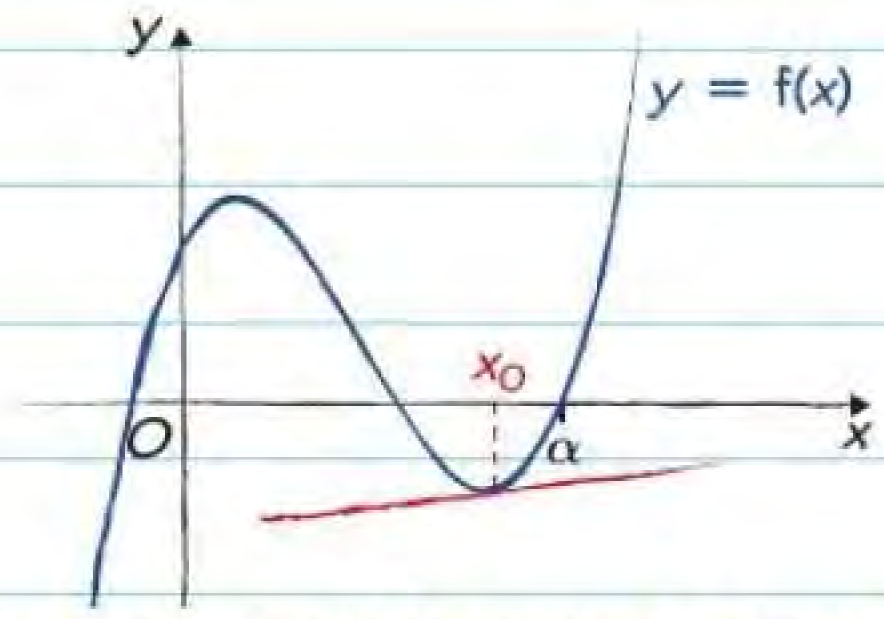
The choice of starting value is important in the Newton–Raphson method. Here, three very similar starting values are used to try and find an approximation to α:



The process will converge on a different root of f(x) = 0.



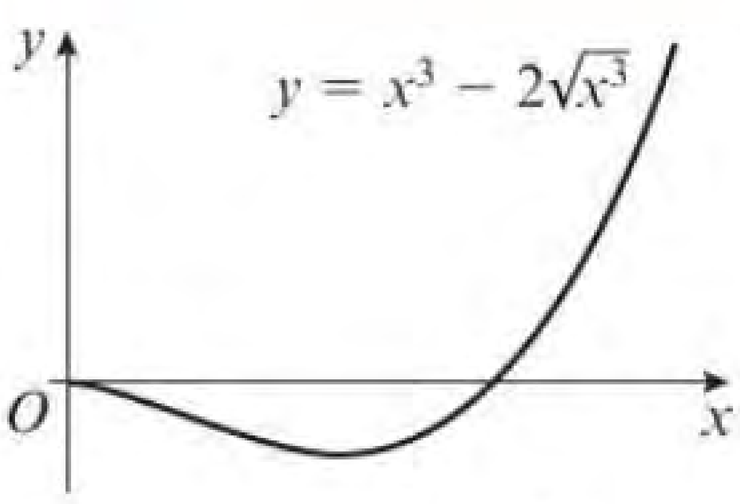
x_0 is a stationary point of the curve, so $f'(x_0) = 0$. The tangent is horizontal and never intersects the x-axis.



The gradient of the tangent is small so the point of intersection is far from α. The process will converge slowly.

Now try this

The diagram shows part of the curve with equation y = f(x) where f(x) = $x^3 - 2\sqrt{x^3}$



(a) Show that f(x) = 0 has a root, α, in the interval [1, 2]. Answer **(2 marks)**

(b) Find $f'(x)$. Answer **(2 marks)**

(c) Explain why $x_0 = 1$ would not be a suitable first approximation when applying the Newton–Raphson procedure to find α. Answer **(1 mark)**

(d) Taking $x_0 = 2$ as a first approximation, apply the Newton–Raphson procedure twice to f(x) to obtain an approximate value of α. Answer **(5 marks)**