Trigonometric Proofs

They say the proof of the pudding is in the eating, but trigonometry ain't no pudding, treacle.

Use the ***Trig Identities*** to prove something is the ***Same*** as something else

As well as simplifying and solving nasty trig equations, you can also use **identities** to **prove** or **'show that'** two **trig expressions** are **the same**. A bit like this, in fact:

Example: Show that $\frac{\cos^2\theta}{1+\sin\theta} \equiv 1 - \sin\theta$.

1) Prove things like this by playing about with one side of the equation until you get the other side.

Left-hand side: $\frac{\cos^2\theta}{1+\sin\theta}$

See page 59.

2) The only thing I can think of doing here is replacing $\cos^2\theta$ with $1 - \sin^2\theta$.
(Which is good because it works.)

$$\equiv \frac{1-\sin^2\theta}{1+\sin\theta}$$

The next trick is the hardest to spot. Look at the top — does that remind you of anything?

3) The top line is a **difference of two squares**:

$1 - a^2 = (1 + a)(1 - a)$
$\Rightarrow 1 - \sin^2\theta = (1 + \sin\theta)(1 - \sin\theta)$

$$\equiv \frac{(1+\sin\theta)(1-\sin\theta)}{1+\sin\theta}$$

$\equiv 1 - \sin\theta$, **the right-hand side.**

This example uses an identity from page 68.

Example: Show that $\frac{\tan^2 x}{\sec x} \equiv \sec x - \cos x$.

Again, take one side of the identity and play about with it until you get the other side:

Left-hand side: $\frac{\tan^2 x}{\sec x}$

Try replacing $\tan^2 x$ with $\sec^2 x - 1$: $\equiv \frac{\sec^2 x - 1}{\sec x} \equiv \frac{\sec^2 x}{\sec x} - \frac{1}{\sec x} \equiv \sec x - \cos x$ ...which is the **right-hand side.**

You might have to use ***Addition*** or ***Double Angle Identities***

You might be asked to use the **addition formulas** to prove an identity (see p.70). Just put the **numbers** and **variables** from the **left-hand side** into the addition formulas and **simplify** until you get the expression you're after.

Example: Prove that $\cos(a + 60°) + \sin(a + 30°) \equiv \cos a$.

Be careful with the + and – signs here

Put the numbers from the question into the **addition formulas**:

$\cos(a + 60°) + \sin(a + 30°) \equiv (\cos a \cos 60° - \sin a \sin 60°) + (\sin a \cos 30° + \cos a \sin 30°)$

Now **substitute** in any sin and cos values that you know... $= \frac{1}{2}\cos a - \frac{\sqrt{3}}{2}\sin a + \frac{\sqrt{3}}{2}\sin a + \frac{1}{2}\cos a$

See p.56.

...and **simplify**: $= \frac{1}{2}\cos a + \frac{1}{2}\cos a = \cos a$

Maths criminals: innocent until proven sin y.

Whenever you have an expression that contains any angle that's **twice the size** of another, you can use the **double angle formulas** (see p.71)— whether it's $\sin x$ and $\sin 2x$, $\cos 2x$ and $\cos 4x$ or $\tan x$ and $\tan \frac{x}{2}$.

This hellish example uses loads of different identities — there are more like this on the next page too. Woohoo.

Example: Prove that $2\left(\cot\frac{x}{2}\right)\left(1-\cos^2\frac{x}{2}\right) \equiv \sin x$.

First, use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to replace $1 - \cos^2\frac{x}{2}$ on the left-hand side: **Left-hand side:** $2\cot\frac{x}{2}\sin^2\frac{x}{2}$

Now write $\cot\theta$ as $\frac{\cos\theta}{\sin\theta}$: $2\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\sin^2\frac{x}{2} \equiv 2\cos\frac{x}{2}\sin\frac{x}{2}$

Now you can use the sin 2A double angle formula to write $\sin x \equiv 2\sin\frac{x}{2}\cos\frac{x}{2}$ (using $A = \frac{x}{2}$).

So using the **sin double angle** formula... $\equiv \sin x$ ...you get the **right-hand side.**

Trigonometric Proofs

Small Angle Approximations can pop up too

Remember those lovely **small angle approximations** from p.69? No?
Well you'd better flick back a few pages for a recap before this example.

This is a whopping clue (pun intended) as to how you're going to tackle this question. If an exam question mentions an angle being 'small', think 'small angle approximations'.

Example: Show that $\frac{2x\sin 2x}{1-\cos 5x} \approx \frac{8}{25}$ when x is small.

Use the **small angle approximations** for each trig function, then **simplify**:

$\sin\theta \approx \theta$ $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$$\frac{2x\sin 2x}{1-\cos 5x} \approx \frac{2x(2x)}{1-\left(1-\frac{1}{2}(5x)^2\right)} = \frac{4x^2}{\frac{25}{2}x^2} = \frac{8}{25}$$

*You might have to use **Different Bits** of **Trig** in the **Same Question***

Some exam questions might try to catch you out by making you use **more than one** identity...

Example: Show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

First, write $\cos 3\theta$ as $\cos(2\theta + \theta)$, then you can use the **cos addition formula**:

$$\cos(3\theta) \equiv \cos(2\theta + \theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$

Now you can use the cos and sin **double angle formulas** to get rid of the 2θ:

$$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta \equiv (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \equiv 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \equiv 4\cos^3\theta - 3\cos\theta$$

You have to use both the addition formula and the double angle formulas in this question.

This uses the identity $\sin^2\theta + \cos^2\theta \equiv 1$ in the form $\sin^2\theta \equiv 1 - \cos^2\theta$.

This next question looks short and sweet, but it's actually pretty nasty
— you need to know a sneaky **conversion** between **sin** and **cos**.

Example: If $y = \arcsin x$ for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, show that $\arccos x = \frac{\pi}{2} - y$.

$y = \arcsin x$, so $x = \sin y$ (as arcsin is the inverse of sin — see p.66).

Now the next bit isn't obvious — you need to use an identity to **switch from sin to cos**. This gives: $x = \cos\left(\frac{\pi}{2} - y\right)$

Now, **taking inverses** gives: $\arccos x = \arccos\left[\cos\left(\frac{\pi}{2} - y\right)\right]$

$$\Rightarrow \arccos x = \frac{\pi}{2} - y$$

Converting sin to cos (and back):
$\sin t \equiv \cos\left(\frac{\pi}{2} - t\right)$
and $\cos t \equiv \sin\left(\frac{\pi}{2} - t\right)$.
Remember sin is just cos shifted by $\frac{\pi}{2}$ and vice versa.

Practice Questions

Q1 Show that $\frac{\sin^4 x + \sin^2 x\cos^2 x}{\cos^2 x - 1} \equiv -1$.

Q2 Use trig identities to show that: a) $\cot^2\theta + \sin^2\theta \equiv \operatorname{cosec}^2\theta - \cos^2\theta$, b) $\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \equiv 2\operatorname{cosec} 2\theta$.

Exam Questions

Q1 a) Show that $\frac{2\sin x}{1-\cos x} - \frac{2\cos x}{\sin x} \equiv 2\operatorname{cosec} x$. [4 marks]

b) Use this result to find all the solutions for which $\frac{2\sin x}{1-\cos x} - \frac{2\cos x}{\sin x} = 4$, $0 < x < 2\pi$. [3 marks]

Q2 Find an approximation for $(4x)^{-1}\operatorname{cosec} 3x\,(2\cos 7x - 2)$, for sufficiently small values of x. [4 marks]

Q3 Prove the identity $\cos\theta\cos 2\theta + \sin\theta\sin 2\theta \equiv \cos\theta$. [4 marks]

Did someone mention treacle pudding...?

And that's your lot for this section. There is nothing left to prove. Except for your well-honed trig skills in the exam...